Group Theory Lecture Notes

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Contents

Chapter 1

Compact simple Lie algebras, roots, and root strings

1.1 Lie algebra and adjoint representation

Let $\mathfrak{g}$ be a compact semisimple Lie algebra of dimension D with a Hermitian basis $\{T^A\}_{A=1}^D$. The commutator defines the structure constants,

$$[T^A, T^B] = \mathrm{i} f^{AB}{}_C T^C. \tag{1.1}$$

The Jacobi identity is

$$f^{AB}{}_D f^{DC}{}_E + f^{BC}{}_D f^{DA}{}_E + f^{CA}{}_D f^{DB}{}_E = 0. \tag{1.2}$$

In the adjoint representation,

$$(T^A_{\mathrm{ad}})^B{}_C = -\mathrm{i} f^{AB}{}_C, \tag{1.3}$$

up to the harmless overall sign convention obtained by defining $\mathrm{ad}(T^A)X = [T^A, X]$ instead.

For a compact algebra one may choose an orthonormal basis with respect to a positive invariant inner product. In such a basis f^{ABC} is totally antisymmetric. This is the condition implicit in the opening page of the manuscript. Total antisymmetry is not a basis-independent statement about arbitrary structure constants.

The invariant bilinear form obtained from the adjoint is the Killing form,

$$B(X, Y) = \mathrm{Tr}_{\mathrm{ad}}(\mathrm{ad}\, X \ \mathrm{ad}\, Y)\,. \tag{1.4}$$

For the compact real form $-B$ is positive definite on a semisimple algebra. It is conventional in physics to absorb the sign and normalization into a positive metric g^{AB}.

1.2 Cartan subalgebra and roots

A Cartan subalgebra $\mathfrak{h}$ of a compact semisimple algebra is a maximal commuting subalgebra. Its dimension is the rank r:

$$[H_i, H_j] = 0, \qquad i, j = 1, \ldots, r. \tag{1.5}$$

After complexification, the adjoint action of all H_i can be diagonalized simultaneously. For each non-zero root $\alpha \in \mathfrak{h}^*$ there is a root vector E_α obeying

$$[H_i, E_\alpha] = \alpha_i E_\alpha. \tag{1.6}$$

For a finite-dimensional complex semisimple algebra every root space is one-dimensional. Hence

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{h}_{\mathbb{C}} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha, \qquad \dim \mathfrak{g} = D = r + |\Phi| \,. \tag{1.7}$$

The compact reality condition permits

$$E_\alpha^\dagger = E_{-\alpha}, \tag{1.8}$$

so roots occur in pairs $\pm\alpha$ and $D - r$ is even.

If $\alpha + \beta$ is a root,

$$[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}. \tag{1.9}$$

If $\alpha + \beta$ is neither a root nor zero, the commutator vanishes. For opposite roots one may normalize

$$[E_\alpha, E_{-\alpha}] = H_\alpha, \qquad H_\alpha \equiv \alpha^\vee \cdot H, \tag{1.10}$$

where $\alpha^\vee = 2\alpha/(\alpha,\alpha)$ is the coroot. In a normalization in which $\mathrm{Tr}(E_\alpha E_{-\alpha}) = 1$, the components of H_α are related to the components of α by the inverse Cartan/Killing metric. This is the distinction emphasized, somewhat cryptically, on source page 6: a root component α_i and a coefficient of H_i are not the same object until the metric is used to identify $\mathfrak{h}^*$ with $\mathfrak{h}$.

The invariant inner product on root space is

$$(\alpha, \beta) = \alpha_i \, g^{ij} \beta_j. \tag{1.11}$$

1.3 Relations among structure constants

With compatible phases for root vectors,

$$N_{\alpha,\beta} = -N_{\beta,\alpha}, \qquad N_{\alpha,\beta} = N_{\beta,-\alpha-\beta} = N_{-\alpha-\beta,\alpha} \tag{1.12}$$

whenever $\alpha+\beta$ is a root. These relations follow from invariance of the bilinear form and the Jacobi identity. Signs may change if the phases of individual E_α are changed; only phase-independent combinations are invariant.

1.4 Root strings

Fix two roots α and β. The roots lying on the line through α in the β direction form a finite string

$$\alpha - p\beta, \ldots, \alpha - \beta, \alpha, \alpha + \beta, \ldots, \alpha + q\beta, \tag{1.13}$$

where $p, q \in \mathbb{Z}_{\geq 0}$ and neither $\alpha - (p+1)\beta$ nor $\alpha + (q+1)\beta$ is a root. The $\mathfrak{su}(2)$ subalgebra generated by $E_{\pm\beta}$ and H_β gives the root-string theorem

$$\boxed{p - q = \frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}.} \tag{1.14}$$

This is the precise version of the recursion derived over source pages 8–15. The formula changes sign if p and q are defined in the opposite order.

The same theorem immediately implies that the Weyl reflection

$$s_\beta(\alpha) = \alpha - \frac{2(\alpha,\beta)}{(\beta,\beta)}\,\beta \tag{1.15}$$

is again a root. It preserves the inner product:

$$(s_\beta\alpha, s_\beta\alpha) = (\alpha, \alpha). \tag{1.16}$$

Applying the root-string integrality condition in both directions gives

$$\frac{4(\alpha,\beta)^2}{(\alpha,\alpha)(\beta,\beta)} = mn, \qquad m, n \in \mathbb{Z}_{\geq 0}, \tag{1.17}$$

and therefore

$$4\cos^2\theta_{\alpha\beta} \in \{0, 1, 2, 3, 4\}. \tag{1.18}$$

For distinct non-proportional roots the value 4 is excluded, leaving $0, 1, 2, 3$. Thus the possible acute or obtuse angles are

$$90^\circ, \quad 60^\circ/120^\circ, \quad 45^\circ/135^\circ, \quad 30^\circ/150^\circ. \tag{1.19}$$

For distinct *simple* roots, $(\alpha_i, \alpha_j) \leq 0$, so only $90^\circ, 120^\circ, 135^\circ, 150^\circ$ occur. The manuscript's list on page 18 gives the three non-orthogonal possibilities; the disconnected 90° case should be added.

1.5 Simple and positive roots

Choose a hyperplane through the origin containing no root. This divides Φ into positive and negative roots,

$$\Phi = \Phi_+ \sqcup (-\Phi_+). \tag{1.20}$$

A positive root is *simple* if it cannot be written as a sum of two positive roots. The set of simple roots

$$\Delta = \{\alpha_1, \ldots, \alpha_r\} \tag{1.21}$$

is linearly independent and forms a basis of the real root space. Every root is an integer linear combination of simple roots with coefficients all nonnegative or all nonpositive:

$$\alpha = \sum_{i=1}^{r} n_i\alpha_i, \qquad n_i \in \mathbb{Z}, \qquad \text{all } n_i \geq 0 \text{ or all } n_i \leq 0. \tag{1.22}$$

For two distinct simple roots, $\alpha_i - \alpha_j$ is not a root, which is one quick way to see

$$(\alpha_i, \alpha_j) \leq 0. \tag{1.23}$$

1.6 The Weyl group

The reflections in the simple roots generate the Weyl group

$$W = \langle s_{\alpha_1}, \ldots, s_{\alpha_r} \rangle. \tag{1.24}$$

It acts by isometries of root and weight space and permutes the complete root system. A choice of positive roots defines a fundamental Weyl chamber. Later, dominant integral weights in this chamber will label finite-dimensional irreducible representations.

Remark 1.1. *The manuscript uses Hermitian generators and often writes roots directly as r-component real vectors. This is legitimate after the invariant form has been used to identify $\mathfrak{h}$ and $\mathfrak{h}^*$. The distinction between a root, a coroot, and the associated Cartan generator becomes essential for B_n, C_n, F_4, G_2.*

Chapter 2

Cartan matrices, Dynkin diagrams, and the classification theorem

2.1 Cartan matrix

The Cartan matrix associated with an ordered set of simple roots is

$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}. \tag{2.1}$$

For a finite-dimensional semisimple Lie algebra it satisfies

1. $A_{ii} = 2$;
2. $A_{ij} \in \mathbb{Z}_{\leq 0}$ for $i \neq j$;
3. $A_{ij} = 0$ iff $A_{ji} = 0$;
4. it is symmetrizable: there is a positive diagonal matrix D such that DA is symmetric;
5. for finite type, the symmetrized matrix is positive definite.

The integer

$$A_{ij}A_{ji} = 4\cos^2\theta_{ij} \tag{2.2}$$

is $0, 1, 2,$ or 3 for $i \neq j$.

A Dynkin diagram has one node for each simple root. Two nodes are joined by $0, 1, 2,$ or 3 bonds according to $A_{ij}A_{ji}$. In a multiple bond, the arrow points toward the *shorter* root. The manuscript instead distinguishes long and short roots by filled and open circles; that conveys the same length data but is not the standard modern convention.

2.2 Why finite Dynkin diagrams are trees

The standard classification argument used in the manuscript can be phrased geometrically. Normalize each simple root to unit length, $u_i = \alpha_i/|\alpha_i|$, so for connected nodes $(u_i, u_j) < 0$. Positivity of the norm of sums of simple roots constrains the total amount of connectivity. In particular, a connected finite-type Dynkin diagram has no closed cycle and no vertex from which more than three bonds emanate. Together with the allowed relative root lengths, these restrictions lead to the finite ADE/BCFG list.

2.3 Classification of compact simple Lie algebras

Up to isomorphism, the complex simple Lie algebras are

$$A_n,\ B_n,\ C_n,\ D_n,\ E_6,\ E_7,\ E_8,\ F_4,\ G_2, \tag{2.3}$$

with

$$A_n \simeq \mathfrak{sl}(n+1,\mathbb{C}), \qquad B_n \simeq \mathfrak{so}(2n+1,\mathbb{C}), \tag{2.4}$$

$$C_n \simeq \mathfrak{sp}(2n,\mathbb{C}), \qquad D_n \simeq \mathfrak{so}(2n,\mathbb{C}). \tag{2.5}$$

The compact real forms are

$$A_n:\ \mathfrak{su}(n+1), \qquad B_n:\ \mathfrak{so}(2n+1), \tag{2.6}$$

$$C_n:\ \mathfrak{usp}(2n), \qquad D_n:\ \mathfrak{so}(2n). \tag{2.7}$$

Low-rank coincidences include

$$A_1 \simeq B_1 \simeq C_1, \qquad D_2 \simeq A_1 \oplus A_1, \qquad D_3 \simeq A_3, \qquad B_2 \simeq C_2. \tag{2.8}$$

For reference, the ranks and dimensions are

algebra	rank	dimension
A_n	n	$n(n+2)$
B_n	n	$n(2n+1)$
C_n	n	$n(2n+1)$
D_n	n	$n(2n-1)$
G_2	2	14
F_4	4	52
E_6	6	78
E_7	7	133
E_8	8	248

2.4 Classical root systems in an orthonormal basis

Let $\{e_i\}$ be an orthonormal basis.

2.4.1 A_n

The A_n root system lives in the n-dimensional hyperplane $\sum_{i=1}^{n+1} x_i = 0$ inside $\mathbb{R}^{n+1}$:

$$\Phi(A_n) = \{e_i - e_j \mid i \neq j,\ 1 \leq i, j \leq n+1\}. \tag{2.9}$$

A conventional simple-root system is

$$\alpha_i = e_i - e_{i+1}, \qquad i = 1, \dots, n. \tag{2.10}$$

All roots have squared length 2. There are $n(n+1)$ roots and $n(n+1)/2$ positive roots.

The highest root is

$$\theta = e_1 - e_{n+1} = \alpha_1 + \cdots + \alpha_n. \tag{2.11}$$

The Weyl vector, by contrast, is

$$\rho = \frac{1}{2}\sum_{\alpha>0} \alpha = \frac{1}{2}\sum_{i=1}^{n+1}(n+2-2i)e_i. \tag{2.12}$$

2.4.2 B_n

For $B_n \simeq \mathfrak{so}(2n+1)$,

$$\Phi(B_n) = \{\pm e_i,\ \pm e_i \pm e_j\ (i<j)\}. \tag{2.13}$$

With $|e_i| = 1$, $\pm e_i$ are short roots and $\pm e_i \pm e_j$ are long roots. One may choose

$$\alpha_i = e_i - e_{i+1} \quad (i = 1, \dots, n-1), \qquad \alpha_n = e_n. \tag{2.14}$$

2.4.3 C_n

For $C_n \simeq \mathfrak{sp}(2n)$,

$$\Phi(C_n) = \{\pm 2e_i,\ \pm e_i \pm e_j\ (i<j)\}, \tag{2.15}$$

with simple roots

$$\alpha_i = e_i - e_{i+1} \quad (i = 1, \dots, n-1), \qquad \alpha_n = 2e_n. \tag{2.16}$$

The B_n and C_n root systems are dual to each other: roots and coroots exchange the long and short systems.

2.4.4 D_n

For $D_n \simeq \mathfrak{so}(2n)$,

$$\Phi(D_n) = \{\pm e_i \pm e_j \mid i<j\}, \tag{2.17}$$

all of one length. A standard simple-root choice is

$$\alpha_i = e_i - e_{i+1} \quad (i = 1, \dots, n-1), \qquad \alpha_n = e_{n-1} + e_n. \tag{2.18}$$

Equivalently, the final two roots can be taken as $e_{n-1} - e_n$ and $e_{n-1} + e_n$, giving the familiar forked D_n diagram.

2.4.5 G_2

A convenient realization lies in the plane $e_1 + e_2 + e_3 = 0$:

$$\Phi(G_2) = \{\pm(e_i - e_j)\} \cup \{\pm(2e_i - e_j - e_k)\}, \qquad i, j, k \text{ all distinct}. \tag{2.19}$$

One simple-root choice is

$$\alpha_1 = e_1 - e_2, \qquad \alpha_2 = -2e_1 + e_2 + e_3. \tag{2.20}$$

Here α_1 is short and α_2 is long. With our Chevalley convention $A_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$,

$$A(G_2) = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}. \tag{2.21}$$

Interchanging the long and short root labels transposes this matrix. This is why both orientations of the triple arrow appear in the literature.

2.4.6 F_4 and the exceptional E-series

The manuscript sketches only selected exceptional simple roots. A complete standard realization of F_4 in $\mathbb{R}^4$ is

$$\Phi(F_4) = \{\pm e_i,\ \pm e_i \pm e_j,\ \tfrac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\}, \tag{2.22}$$

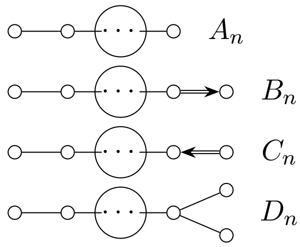
where all 16 sign choices occur in the half-integer set. A common simple system is

$$\alpha_1 = e_2 - e_3, \quad \alpha_2 = e_3 - e_4, \quad \alpha_3 = e_4, \quad \alpha_4 = \tfrac{1}{2}(e_1 - e_2 - e_3 - e_4). \tag{2.23}$$

The E_6, E_7, E_8 systems may be embedded in the E_8 lattice. For most computations the Dynkin diagram and Cartan matrix are more economical than an explicit Euclidean realization.

2.5 Dynkin diagrams

We use the following compact notation for the classical families:



Deleting nodes from a Dynkin diagram produces root subsystems and therefore regular subalgebras, but the precise maximal-subalgebra statement requires the extended-diagram discussion of Chapter 6. The manuscript's early node-deletion statement should be read in this qualified sense.

Chapter 3

Finite-dimensional representation theory in the Dynkin basis

3.1 Weights and Dynkin labels

Let V be a finite-dimensional representation of $\mathfrak{g}$. A weight state $|\lambda\rangle$ is a simultaneous eigenstate of the Cartan generators. Its Dynkin labels are

$$a_i(\lambda) = \frac{2(\lambda, \alpha_i)}{(\alpha_i, \alpha_i)} \in \mathbb{Z} \tag{3.1}$$

for weights of an integrable finite-dimensional representation. A weight is *dominant* if $a_i \geq 0$ for all i, and *dominant integral* if, in addition, all a_i are integers.

The fundamental weights ω_i are defined by

$$\frac{2(\omega_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij}. \tag{3.2}$$

Thus a dominant highest weight is

$$\Lambda = \sum_{i=1}^{r} a_i \omega_i, \qquad a_i \in \mathbb{Z}_{\geq 0}. \tag{3.3}$$

Finite-dimensional irreducible representations are in one-to-one correspondence with dominant integral highest weights.

3.2 Chevalley raising and lowering operators

For each simple root introduce Chevalley generators

$$e_i = E_{\alpha_i}, \qquad f_i = E_{-\alpha_i}, \qquad h_i = H_{\alpha_i}. \tag{3.4}$$

They obey

$$[e_i, f_j] = \delta_{ij} h_i, \qquad [h_i, e_j] = A_{ij} e_j, \qquad [h_i, f_j] = -A_{ij} f_j. \tag{3.5}$$

The triple (e_i, f_i, h_i) is an $\mathfrak{sl}(2)$ subalgebra. If

$$h_i \, |\lambda\rangle = a_i \, |\lambda\rangle \,, \tag{3.6}$$

then f_i lowers the weight by α_i:

$$f_i |\lambda\rangle \propto |\lambda - \alpha_i\rangle . \tag{3.7}$$

A highest-weight state obeys

$$e_i |\Lambda\rangle = 0, \qquad i = 1, \dots, r. \tag{3.8}$$

Starting from $|\Lambda\rangle$ and applying lowering operators produces the complete irreducible representation after the appropriate null states are quotiented out.

3.3 Simple roots in Dynkin coordinates

Because

$$a_i(\alpha_j) = \frac{2(\alpha_j, \alpha_i)}{(\alpha_i, \alpha_i)} = A_{ij}, \tag{3.9}$$

the Dynkin-label vector of the simple root α_j is the jth *column* of A in the convention used here. Therefore lowering a weight by α_j subtracts the jth column. In simply-laced examples such as A_3 the Cartan matrix is symmetric, so the row/column distinction is invisible; it is essential for B_n, C_n, F_4, G_2.

3.4 A useful level vector

The manuscript introduces

$$R_i = 2 \sum_j (A^{-1})_{ij}. \tag{3.10}$$

This vector has a clean interpretation. Since

$$2\rho = \sum_i R_i \alpha_i, \tag{3.11}$$

R_i are the simple-root coefficients of 2ρ. For a highest weight Λ the height difference between the highest and lowest weights is controlled by the Weyl action and, for a fundamental representation, the numbers in (3.10) reproduce the maximal level shown in the manuscript's A_3 examples. It is better to call R the coefficient vector of 2ρ rather than a universal "number of lowering operations", because multiple lowering paths and multiplicities can occur.

3.5 Example: $A_3 \simeq \mathfrak{su}(4)$

The Cartan matrix is

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}, \qquad A^{-1} = \begin{pmatrix} \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \end{pmatrix}. \tag{3.12}$$

Therefore

$$R = (3, 4, 3). \tag{3.13}$$

The three fundamental highest weights have Dynkin labels

$$(1, 0, 0), \qquad (0, 1, 0), \qquad (0, 0, 1). \tag{3.14}$$

The first generates the **4**:

$$(1,0,0) \xrightarrow{-\alpha_1} (-1,1,0) \xrightarrow{-\alpha_2} (0,-1,1) \xrightarrow{-\alpha_3} (0,0,-1). \tag{3.15}$$

The third is the conjugate $\bar{\mathbf{4}}$ and has the reversed weight tree. The middle fundamental representation is the antisymmetric two-index tensor **6**:

$$\mathbf{6} = \wedge^2 \mathbf{4}, \tag{3.16}$$

and it is self-conjugate.

For A_n, complex conjugation reverses the Dynkin labels:

$$(a_1, a_2, \ldots, a_n)^* = (a_n, a_{n-1}, \ldots, a_1). \tag{3.17}$$

Thus the two end-node fundamental representations of A_3 are conjugate and the middle node is self-conjugate, exactly as the source weight trees intend to illustrate.

3.6 Tensor representations of $SU(N)$

For $SU(N)$ the fundamental weights correspond to antisymmetric tensors:

$$\omega_k \longleftrightarrow \wedge^k \mathbf{N}, \qquad \dim \wedge^k \mathbf{N} = \binom{N}{k}. \tag{3.18}$$

A general irreducible representation can equivalently be described by a Young diagram. If the Dynkin labels are $(a_1, \ldots, a_{N-1})$, the row lengths are

$$\ell_i = \sum_{j=i}^{N-1} a_j, \qquad i = 1, \ldots, N-1. \tag{3.19}$$

The converse is $a_i = \ell_i - \ell_{i+1}$.

3.7 Weight multiplicities

A weight may occur more than once. Weight-tree sketches that show only nodes and arrows are exact for multiplicity-free examples such as the fundamental **4** of $SU(4)$, but a general representation requires a multiplicity formula. Later we will derive this information from the Weyl character formula and, at the Verma-module level, from the Kostant partition function.

Chapter 4

Orthogonal algebras, Clifford algebras, and spinor representations

4.1 Orthogonal versus unitary algebras

The Lie algebra $\mathfrak{so}(N)$ consists of infinitesimal rotations preserving a real Euclidean norm. In the defining representation one may take Hermitian generators

$$X_{ij} = \mathrm{i}\left(x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}\right), \qquad X_{ij} = -X_{ji}, \tag{4.1}$$

with

$$[X_{ij}, X_{k\ell}] = \mathrm{i}\left(\delta_{ik}X_{j\ell} - \delta_{i\ell}X_{jk} - \delta_{jk}X_{i\ell} + \delta_{j\ell}X_{ik}\right). \tag{4.2}$$

For $N = 2n$, the n commuting rotations

$$X_{12}, X_{34}, \ldots, X_{2n-1,2n} \tag{4.3}$$

form a Cartan subalgebra, hence

$$\operatorname{rank}\mathfrak{so}(2n) = n. \tag{4.4}$$

For $N = 2n+1$ no additional commuting independent rotation can be added, so

$$\operatorname{rank}\mathfrak{so}(2n+1) = n. \tag{4.5}$$

At Lie-algebra level there are important low-dimensional isomorphisms,

$$\mathfrak{so}(3) \simeq \mathfrak{su}(2), \qquad \mathfrak{so}(4) \simeq \mathfrak{su}(2) \oplus \mathfrak{su}(2), \qquad \mathfrak{so}(6) \simeq \mathfrak{su}(4). \tag{4.6}$$

At group level the simply connected covers are

$$\mathrm{Spin}(3) \simeq SU(2), \qquad \mathrm{Spin}(4) \simeq SU(2) \times SU(2), \qquad \mathrm{Spin}(6) \simeq SU(4), \tag{4.7}$$

whereas $SO(N)$ itself differs by a discrete quotient. This global distinction is important when spinor representations are discussed.

4.2 From Pauli matrices to Clifford algebras

The Pauli matrices satisfy

$$\sigma_a\sigma_b = \delta_{ab}\mathbf{1} + \mathrm{i}\epsilon_{abc}\sigma_c, \qquad \{\sigma_a, \sigma_b\} = 2\delta_{ab}\mathbf{1}. \tag{4.8}$$

Thus they provide a representation of the Euclidean Clifford algebra in three dimensions. Given Clifford matrices

$$\{\Gamma_a, \Gamma_b\} = 2\delta_{ab}\mathbf{1}, \tag{4.9}$$

the spin generators

$$\Sigma_{ab} = -\frac{\mathrm{i}}{4}[\Gamma_a, \Gamma_b] \tag{4.10}$$

satisfy the $\mathfrak{so}(N)$ commutation relations. The manuscript frequently uses $X_{ab} = -\frac{\mathrm{i}}{2}\Gamma_a\Gamma_b$ for $a \neq b$, which is the same expression because $\Gamma_a\Gamma_b = -\Gamma_b\Gamma_a$ when $a \neq b$.

4.3 Fermionic oscillator construction of $\mathrm{Spin}(2n)$ spinors

Introduce n fermionic oscillators

$$\{b_i, b_j^\dagger\} = \delta_{ij}, \qquad \{b_i, b_j\} = \{b_i^\dagger, b_j^\dagger\} = 0. \tag{4.11}$$

The Fock space is

$$\mathcal{S} = \bigoplus_{k=0}^{n} \wedge^k \mathbb{C}^n, \qquad \dim \mathcal{S} = 2^n. \tag{4.12}$$

Under the natural $U(n) \subset SO(2n)$ subgroup the number-preserving bilinears

$$T^i{}_j = b_i^\dagger b_j - \frac{1}{2}\delta^i{}_j \tag{4.13}$$

generate $\mathfrak{u}(1) \oplus \mathfrak{su}(n)$, while

$$A^{ij} = b_i^\dagger b_j^\dagger, \qquad A_{ij} = b_i b_j \tag{4.14}$$

supply the remaining generators. Together they close on $\mathfrak{so}(2n)$:

$$\mathfrak{so}(2n) \supset \mathfrak{su}(n) \oplus \mathfrak{u}(1). \tag{4.15}$$

The dimension check is

$$n^2 + n(n-1) = n(2n-1) = \dim \mathfrak{so}(2n). \tag{4.16}$$

Fermion parity separates the Dirac spinor into the two chiral spinors,

$$\mathcal{S}_+ = \bigoplus_{k \text{ even}} \wedge^k \mathbb{C}^n, \qquad \mathcal{S}_- = \bigoplus_{k \text{ odd}} \wedge^k \mathbb{C}^n, \tag{4.17}$$

each of dimension

$$\dim \mathcal{S}_\pm = 2^{n-1}. \tag{4.18}$$

This is the precise oscillator construction intended on source pages 41–46.

4.3.1 Example: $\mathrm{Spin}(6) \simeq SU(4)$

For $n = 3$,

$$\mathcal{S}_+ = \wedge^0\mathbf{3} \oplus \wedge^2\mathbf{3} = \mathbf{1} \oplus \bar{\mathbf{3}}, \tag{4.19}$$

$$\mathcal{S}_- = \wedge^1\mathbf{3} \oplus \wedge^3\mathbf{3} = \mathbf{3} \oplus \mathbf{1}. \tag{4.20}$$

Under the standard $SU(3) \times U(1) \subset SU(4)$ embedding these are, up to the choice of chirality and the sign of the $U(1)$ generator, the branching patterns of $\bar{\mathbf{4}}$ and $\mathbf{4}$. Thus the two chiral spinors are complex conjugates, agreeing with $\mathrm{Spin}(6) \simeq SU(4)$.

4.3.2 Example: Spin(8) and triality

For $n = 4$,

$$\dim S_+ = \dim S_- = 8, \tag{4.21}$$

giving 8_s and 8_c. The vector 8_v has the same dimension. The outer automorphism group S_3 of the D_4 Dynkin diagram permutes

$$8_v, \qquad 8_s, \qquad 8_c, \tag{4.22}$$

which is the triality symmetry implicit in several later embedding examples.

4.4 Reality properties of spinors

For $\mathrm{Spin}(2n)$ chiral spinors:

- if n is odd, $S_+^* \simeq S_-$, so the two chiral spinors are complex conjugates;
- if $n \equiv 0 \pmod 4$, each $S_\pm$ is real;
- if $n \equiv 2 \pmod 4$, each $S_\pm$ is pseudoreal.

Examples are

$$\mathrm{Spin}(6): \; 4, \bar{4} \text{ complex}; \qquad \mathrm{Spin}(8): \; 8_s, 8_c \text{ real}; \qquad \mathrm{Spin}(10): \; 16, \overline{16} \text{ complex}. \tag{4.23}$$

The analogous statement for $\mathrm{Spin}(2n+1)$ follows the familiar mod-8 periodicity of real Clifford algebras. The spinor dimension is always 2^n.

4.5 Dynkin labels and tensor representations of orthogonal groups

For $D_n = \mathfrak{so}(2n)$, the two end nodes of the fork correspond to the chiral spinors

$$S_+: \; (0, \dots, 0, 1, 0), \qquad S_-: \; (0, \dots, 0, 0, 1), \tag{4.24}$$

each of dimension 2^{n-1}. The vector is $(1, 0, \dots, 0)$ with dimension $2n$. For $1 \le k \le n-2$, the kth fundamental representation is the antisymmetric tensor $\wedge^k(\mathbf{2n})$ with dimension $\binom{2n}{k}$.

For $B_n = \mathfrak{so}(2n+1)$, the final node is the spinor $(0, \dots, 0, 1)$ of dimension 2^n; the earlier nodes give antisymmetric tensors of the vector.

4.6 Recursive construction of gamma matrices

Suppose $\Gamma_1, \dots, \Gamma_m$ obey the m-dimensional Euclidean Clifford algebra. A standard doubling construction is

$$\Gamma'_a = \sigma_1 \otimes \Gamma_a, \qquad a = 1, \dots, m, \tag{4.25}$$
$$\Gamma'_{m+1} = \sigma_2 \otimes \mathbf{1}, \tag{4.26}$$
$$\Gamma'_{m+2} = \sigma_3 \otimes \mathbf{1}. \tag{4.27}$$

Then

$$\{\Gamma'_A, \Gamma'_B\} = 2\delta_{AB}. \tag{4.28}$$

This provides a clean recursive version of the tensor-product Pauli-matrix construction worked out on source pages 54–62.

For even $d = 2n$ the chirality matrix can be chosen as

$$\Gamma_* = \mathrm{i}^{-n}\Gamma_1\Gamma_2\cdots\Gamma_{2n}, \qquad \Gamma_*^2 = \mathbf{1}, \qquad \{\Gamma_*, \Gamma_a\} = 0, \tag{4.29}$$

with projectors

$$P_\pm = \frac{1}{2}(1 \pm \Gamma_*). \tag{4.30}$$

4.7 Charge conjugation

A charge-conjugation matrix C is defined, up to convention-dependent signs, by

$$C\Gamma_a C^{-1} = \eta_C\,\Gamma_a^T, \qquad \eta_C = \pm 1. \tag{4.31}$$

Its existence and symmetry $C^T = \pm C$ depend on the dimension modulo 8. This is the systematic framework behind the direct matrix manipulations on source pages 59–62. A representation is

- *real* if there exists a symmetric intertwiner relating it to its complex conjugate;
- *pseudoreal* if the intertwiner is antisymmetric;
- *complex* if no such intertwiner exists.

4.8 Products of spinors and differential forms

Clifford multiplication gives the Fierz-theoretic relation between spinor bilinears and antisymmetric tensors. Schematically,

$$S \otimes S \simeq \bigoplus_p \wedge^p V, \tag{4.32}$$

with the allowed parity and self-duality determined by dimension and chirality.

For $\mathrm{Spin}(6) \simeq SU(4)$,

$$\mathbf{4} \otimes \bar{\mathbf{4}} = \mathbf{1} \oplus \mathbf{15}, \tag{4.33}$$

$$\mathbf{4} \otimes \mathbf{4} = \mathbf{6} \oplus \mathbf{10}. \tag{4.34}$$

The $\mathbf{6}$ is the vector of $SO(6)$, the $\mathbf{15}$ is the adjoint, and the $\mathbf{10}$ is one complex self-duality sector of a three-form. Since a three-form in six dimensions has $\binom{6}{3} = 20$ components and $\star^2 = -1$ in Euclidean signature on three-forms, the complexified space splits into $\pm\mathrm{i}$ eigenspaces:

$$\mathbf{20}_{\mathbb{C}} = \mathbf{10} \oplus \overline{\mathbf{10}}. \tag{4.35}$$

This corrects the implicit assumption in the manuscript that the relevant six-dimensional three-form self-duality eigenvalues are simply ± 1 in Euclidean signature.

Chapter 5

Casimir invariants, Dynkin indices, congruence classes, and birdtracks

5.1 Casimir operators

Let $\{T^A\}$ be generators of a simple Lie algebra $\mathfrak{g}$ in a representation R, with

$$[T^A, T^B] = \mathrm{i} f^{ABC} T^C. \tag{5.1}$$

A Casimir element is an element of the center of the universal enveloping algebra $U(\mathfrak{g})$. In particular, after choosing an orthonormal basis for the invariant bilinear form, the quadratic Casimir is

$$C_2(R) = T_R^A T_R^A, \qquad [C_2(R), T_R^B] = 0. \tag{5.2}$$

Schur's lemma therefore implies, on an irreducible representation,

$$T_R^A T_R^A = C_2(R)\, \mathbf{1}_R. \tag{5.3}$$

For a rank-r simple Lie algebra there are r algebraically independent primitive invariant polynomials. Their degrees are the exponents of $\mathfrak{g}$ plus one. The complete list is

Algebra	Degrees of primitive invariants
A_n	$2, 3, \ldots, n+1$
B_n	$2, 4, 6, \ldots, 2n$
C_n	$2, 4, 6, \ldots, 2n$
D_n	$2, 4, 6, \ldots, 2n-2, n$
G_2	$2, 6$
F_4	$2, 6, 8, 12$
E_6	$2, 5, 6, 8, 9, 12$
E_7	$2, 6, 8, 10, 12, 14, 18$
E_8	$2, 8, 12, 14, 18, 20, 24, 30$

5.2 Quadratic Dynkin index and its relation to the Casimir

For a representation R define the quadratic Dynkin index $T(R)$ by

$$\mathrm{Tr}_R(T^A T^B) = T(R)\, \delta^{AB}. \tag{5.4}$$

Taking $A = B$ and summing over the adjoint index gives

$$T(R)\,\dim\mathfrak{g} = \mathrm{Tr}_R(T^A T^A) = C_2(R)\,\dim R, \tag{5.5}$$

so that

$$\boxed{T(R) = \frac{\dim R}{\dim\mathfrak{g}}\,C_2(R).} \tag{5.6}$$

The normalization is conventional. For $SU(N)$ we shall use

$$\mathrm{Tr}_{\mathbf{N}}(T^a T^b) = \frac{1}{2}\delta^{ab}, \tag{5.7}$$

which yields

$$T(\mathbf{N}) = \frac{1}{2}, \qquad C_F = \frac{N^2-1}{2N}, \qquad T(\mathrm{Adj}) = C_A = N. \tag{5.8}$$

For a highest weight λ and the canonical Lie-theory inner product in which long roots have length squared 2, the quadratic-Casimir eigenvalue is

$$\boxed{C_{2,\mathrm{can}}(\lambda) = (\lambda, \lambda + 2\rho).} \tag{5.9}$$

Generator normalizations rescale this value. In particular, for $SU(N)$ with the particle-physics convention $\mathrm{Tr}_{\mathbf{N}}(T^a T^b) = \frac{1}{2}\delta^{ab}$,

$$C_{2,\mathrm{phys}}(R) = \frac{1}{2}C_{2,\mathrm{can}}(R), \tag{5.10}$$

so the fundamental gives $(N^2-1)/(2N)$. The quadratic index $T(R)$ rescales in the same way.

5.3 Cubic invariant and anomaly index

For $SU(N)$ with $N \geq 3$ there is a nonzero symmetric invariant tensor d^{abc}. One may define a cubic index $A(R)$ by

$$\mathrm{Tr}_R\Big(T^a\{T^b, T^c\}\Big) = A(R)\,d^{abc}, \tag{5.11}$$

with a chosen normalization of d^{abc}. Equivalently, the symmetric part of $\mathrm{Tr}_R(T^a T^b T^c)$ carries the cubic invariant, whereas the antisymmetric part is fixed by the structure constants. The existence of a primitive cubic invariant is precisely the degree-3 entry in the A_n series.

5.4 Additivity and tensor-product identities

The quadratic index satisfies

$$T(R_1 \oplus R_2) = T(R_1) + T(R_2), \tag{5.12}$$
$$T(R_1 \otimes R_2) = \dim(R_2)T(R_1) + \dim(R_1)T(R_2). \tag{5.13}$$

The second relation follows by writing

$$T^a_{R_1\otimes R_2} = T^a_{R_1} \otimes \mathbf{1} + \mathbf{1} \otimes T^a_{R_2} \tag{5.14}$$

and observing that the mixed trace vanishes for generators of a simple algebra. Similarly, for a cubic anomaly coefficient,

$$A(R_1 \oplus R_2) = A(R_1) + A(R_2), \tag{5.15}$$

and for a direct product representation of a fixed simple factor,

$$A(R_1 \otimes R_2) = \dim(R_2)A(R_1) + \dim(R_1)A(R_2). \tag{5.16}$$

For a conjugate representation,

$$A(\bar{R}) = -A(R), \tag{5.17}$$

while real and pseudoreal representations have vanishing perturbative cubic gauge anomaly.

5.5 Congruence classes and center charge

Weights fall into congruence classes determined by the finite abelian group

$$P/Q, \tag{5.18}$$

where P is the weight lattice and Q the root lattice. For the simply connected group this is isomorphic to its center. A representation has a definite center character, and tensor products add these charges.

For $SU(N)$, with Dynkin labels $\lambda = (a_1, \ldots, a_{N-1})$, the N-ality is

$$\boxed{q(\lambda) = \sum_{j=1}^{N-1} j\, a_j \pmod{N}.} \tag{5.19}$$

This is equivalently the number of boxes in the Young diagram modulo N.

5.6 Birdtrack notation

The manuscript introduces graphical representation theory ("birdtracks") as a compact bookkeeping language for invariant tensors. A line denotes an index, vertices denote invariant tensors or generators, joining lines denotes index contraction, and closed loops produce dimensions. For instance, in the fundamental of $SU(N)$, with $\mathrm{Tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$,

$$(T^a)_i{}^j (T^a)_k{}^l = \frac{1}{2}\left(\delta_i{}^l \delta_k{}^j - \frac{1}{N}\delta_i{}^j \delta_k{}^l\right). \tag{5.20}$$

Graphically this is the completeness relation that resolves an adjoint line into fundamental-index flow. It will reappear in the color decomposition of gauge amplitudes.

Chapter 6

Embeddings of Lie algebras and extended Dynkin diagrams

6.1 What an embedding means

An embedding $\mathfrak{h} \hookrightarrow \mathfrak{g}$ is an injective Lie-algebra homomorphism. Once an embedding is specified, every representation of $\mathfrak{g}$ branches into representations of $\mathfrak{h}$. In particular,

$$\mathrm{Adj}_{\mathfrak{g}} \downarrow_{\mathfrak{h}} = \mathrm{Adj}_{\mathfrak{h}} \oplus \bigoplus_i R_i, \tag{6.1}$$

where the additional R_i describe the complement of $\mathfrak{h}$ inside $\mathfrak{g}$ as an $\mathfrak{h}$-module.

The notes formulate several practical "rules" based on Dynkin diagrams and branching. These are useful, but they should not be treated as a proof of every possible maximal embedding.

6.2 Regular subalgebras from Dynkin diagrams

Deleting an ordinary node from a finite Dynkin diagram leaves a diagram whose connected components define a regular semisimple subalgebra of rank $r-1$. Depending on the desired rank and centralizer, one often obtains a reductive subalgebra with an additional $\mathfrak{u}(1)$ factor. For example,

$$\mathfrak{su}(N) \supset \mathfrak{su}(p) \oplus \mathfrak{su}(N-p) \oplus \mathfrak{u}(1). \tag{6.2}$$

The fundamental branches as

$$\mathbf{N} \longrightarrow (\mathbf{p}, \mathbf{1})_{q_1} \oplus (\mathbf{1}, \mathbf{N-p})_{q_2}, \tag{6.3}$$

with $pq_1 + (N-p)q_2 = 0$.

6.3 The $SU(3)$ example and inequivalent $SU(2)$ embeddings

For the regular embedding $SU(2) \times U(1) \subset SU(3)$,

$$\mathbf{3} \to \mathbf{2}_q \oplus \mathbf{1}_{-2q}, \tag{6.4}$$

and

$$\mathbf{8} \to \mathbf{3}_0 \oplus \mathbf{2}_{3q} \oplus \mathbf{2}_{-3q} \oplus \mathbf{1}_0. \tag{6.5}$$

There is also a principal $SU(2)$ embedding, inequivalent to the regular one, for which

$$\mathbf{8} \to \mathbf{3} \oplus \mathbf{5}. \tag{6.6}$$

This is a useful warning: the same abstract subalgebra may be embedded in different ways, producing different branching rules.

6.4 $SO(8)$ and triality

The algebra $D_4 = \mathfrak{so}(8)$ has an outer automorphism group S_3 permuting its three outer Dynkin nodes. Consequently the vector and two chiral spinors,

$$\mathbf{8}_v, \qquad \mathbf{8}_s, \qquad \mathbf{8}_c, \tag{6.7}$$

are permuted by triality. This explains why apparently different embeddings and branching rules in the manuscript can be related by an outer automorphism. A standard maximal-rank branching is

$$\mathfrak{so}(8) \supset \mathfrak{so}(4) \oplus \mathfrak{so}(4) \simeq \mathfrak{su}(2)^4. \tag{6.8}$$

Another important one is

$$\mathfrak{so}(8) \supset \mathfrak{su}(4) \oplus \mathfrak{u}(1). \tag{6.9}$$

6.5 Extended Dynkin diagrams and highest roots

Let the highest root be

$$\theta = \sum_{i=1}^{r} a_i \alpha_i, \tag{6.10}$$

where a_i are the marks. Introduce the affine root

$$\alpha_0 = -\theta. \tag{6.11}$$

Adding the node α_0 gives the extended (affine) Dynkin diagram. Removing a node from the extended diagram is a powerful way of finding maximal regular subalgebras of equal rank.

For A_{N-1},

$$\theta = \alpha_1 + \cdots + \alpha_{N-1}, \tag{6.12}$$

so all marks are one. For D_4 the highest root is

$$\theta = \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 \tag{6.13}$$

for the conventional central node α_2. For G_2, with α_1 short and α_2 long,

$$\theta = 3\alpha_1 + 2\alpha_2. \tag{6.14}$$

6.6 Embedding index

If $\mathfrak{h} \hookrightarrow \mathfrak{g}$ are simple and the invariant forms are normalized, the embedding index $I_{\mathfrak{h}\hookrightarrow\mathfrak{g}}$ compares the restrictions of the forms:

$$(X, Y)_{\mathfrak{g}} = I_{\mathfrak{h}\hookrightarrow\mathfrak{g}} (X, Y)_{\mathfrak{h}}, \qquad X, Y \in \mathfrak{h}. \tag{6.15}$$

Equivalently, it may be inferred from the Dynkin indices of the representations appearing when a representation of $\mathfrak{g}$ is restricted to $\mathfrak{h}$. This turns the branching checks in the manuscript into a quantitative consistency test.

Chapter 7

Verma modules, Kostant multiplicities, and Weyl formulae

7.1 Triangular decomposition and highest-weight modules

For a complex semisimple Lie algebra,

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+. \tag{7.1}$$

A highest-weight vector v_λ obeys

$$e_i v_\lambda = 0, \qquad h_i v_\lambda = \lambda(h_i) v_\lambda. \tag{7.2}$$

The Verma module is

$$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b}_+)} \mathbb{C}_\lambda \simeq U(\mathfrak{n}_-) v_\lambda. \tag{7.3}$$

By the Poincaré–Birkhoff–Witt theorem, ordered products of lowering operators span $M(\lambda)$.

A finite-dimensional irreducible representation is generally not the Verma module itself; it is the irreducible quotient

$$L(\lambda) = M(\lambda)/N(\lambda), \tag{7.4}$$

where $N(\lambda)$ is the maximal proper submodule generated by singular vectors.

7.2 $\mathfrak{sl}_2$ example

Let $[h, e] = 2e$, $[h, f] = -2f$, $[e, f] = h$, and let

$$h v_\lambda = \lambda v_\lambda, \qquad e v_\lambda = 0. \tag{7.5}$$

Then

$$e f^m v_\lambda = m(\lambda - m + 1) f^{m-1} v_\lambda. \tag{7.6}$$

For $\lambda = n \in \mathbb{Z}_{\geq 0}$, the vector

$$f^{n+1} v_n \tag{7.7}$$

is singular and generates a Verma submodule isomorphic to $M(-n-2)$. Therefore

$$L(n) = M(n)/M(-n-2), \qquad \dim L(n) = n + 1. \tag{7.8}$$

For the example $n = 2$ in the notes,

$$\operatorname{ch} L(2) = e^2 + 1 + e^{-2}. \tag{7.9}$$

7.3 Kostant partition function

The Kostant partition function $P(\beta)$ counts the number of ways of writing a weight β as a nonnegative integer sum of positive roots. Its generating function is

$$\boxed{\prod_{\alpha\in\Phi_+}\frac{1}{1-e^{-\alpha}}=\sum_{\beta\in Q_+}P(\beta)e^{-\beta}.} \tag{7.10}$$

Accordingly,

$$\operatorname{ch}M(\lambda)=\frac{e^{\lambda}}{\prod_{\alpha>0}(1-e^{-\alpha})}. \tag{7.11}$$

For a finite-dimensional highest-weight representation, Kostant's multiplicity formula is

$$\boxed{m_\lambda(\mu)=\sum_{w\in W}(-1)^{\ell(w)}P(w(\lambda+\rho)-(\mu+\rho))\,.} \tag{7.12}$$

This is the systematic version of the "Kostant counter" construction in the manuscript.

7.4 Weyl denominator and character formula

The Weyl denominator identity is

$$\sum_{w\in W}(-1)^{\ell(w)}e^{w(\rho)}=e^{\rho}\prod_{\alpha>0}(1-e^{-\alpha}). \tag{7.13}$$

The Weyl character formula follows:

$$\boxed{\chi_\lambda=\frac{\displaystyle\sum_{w\in W}(-1)^{\ell(w)}e^{w(\lambda+\rho)}}{\displaystyle\sum_{w\in W}(-1)^{\ell(w)}e^{w(\rho)}}.} \tag{7.14}$$

Equivalently,

$$\chi_\lambda=e^{-\rho}\frac{\sum_w(-1)^{\ell(w)}e^{w(\lambda+\rho)}}{\prod_{\alpha>0}(1-e^{-\alpha})}. \tag{7.15}$$

7.5 Weyl dimension formula

Taking the identity element of the maximal torus gives

$$\boxed{\dim L(\lambda)=\prod_{\alpha>0}\frac{(\lambda+\rho,\alpha)}{(\rho,\alpha)}.} \tag{7.16}$$

For $SU(N)$, if the corresponding Young diagram has nonincreasing row lengths $r_1\geq\cdots\geq r_N=0$, this becomes

$$\dim R=\prod_{1\leq i<j\leq N}\frac{r_i-r_j+j-i}{j-i}. \tag{7.17}$$

For $SU(4)$ with Dynkin labels (a, b, c), the row lengths are

$$r_1 = a + b + c, \qquad r_2 = b + c, \qquad r_3 = c, \qquad r_4 = 0, \tag{7.18}$$

and hence

$$\boxed{\dim(a, b, c) = \frac{(a+1)(b+1)(c+1)(a+b+2)(b+c+2)(a+b+c+3)}{12}.} \tag{7.19}$$

7.6 Quadratic Casimir from the shifted weight

The combination that repeatedly appears in the notes is

$$C_2(\lambda) = (\lambda + \rho, \lambda + \rho) - (\rho, \rho). \tag{7.20}$$

This is identical to $(\lambda, \lambda + 2\rho)$. It is Weyl invariant under the shifted action $\lambda + \rho \mapsto w(\lambda + \rho)$, which explains why shifted Weyl reflections organize singular vectors and character subtractions.

Chapter 8

Serre presentation and affine Kac–Moody algebras

8.1 Chevalley–Serre presentation

Let $A = (A_{ij})$ be a finite Cartan matrix. The corresponding semisimple algebra is generated by e_i, f_i, h_i, $i = 1, \ldots, r$, with

$$[h_i, h_j] = 0, \tag{8.1}$$
$$[h_i, e_j] = A_{ij} e_j, \tag{8.2}$$
$$[h_i, f_j] = -A_{ij} f_j, \tag{8.3}$$
$$[e_i, f_j] = \delta_{ij} h_i, \tag{8.4}$$

and the Serre relations

$$(\mathrm{ad}\, e_i)^{1-A_{ij}} e_j = 0, \qquad i \neq j, \tag{8.5}$$
$$(\mathrm{ad}\, f_i)^{1-A_{ij}} f_j = 0, \qquad i \neq j. \tag{8.6}$$

For example, a single link has $A_{ij} = -1$ and therefore $(\mathrm{ad}\, e_i)^2 e_j = 0$; a triple link in G_2 produces an exponent 4 in the appropriate direction.

8.2 Loop algebra and central extension

Starting with a finite-dimensional simple algebra $\mathfrak{g}$, the loop algebra is

$$L\mathfrak{g} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]. \tag{8.7}$$

Writing $T^a_m = T^a \otimes t^m$,

$$[T^a_m, T^b_n] = \mathrm{i} f^{ab}{}_c T^c_{m+n}. \tag{8.8}$$

Its universal central extension gives the untwisted affine Kac–Moody algebra,

$$\boxed{[T^a_m, T^b_n] = \mathrm{i} f^{ab}{}_c T^c_{m+n} + m\, \kappa^{ab} \delta_{m+n,0} K,} \tag{8.9}$$

with central generator K. Adding the derivation d,

$$[d, T^a_m] = m T^a_m, \qquad [d, K] = 0. \tag{8.10}$$

In a representation, the central element acts as

$$K = k\, \mathbf{1}, \tag{8.11}$$

and k is the level.

8.3 Affine roots, marks, and comarks

The affine simple roots are

$$\alpha_0 = \delta - \theta, \qquad \alpha_1, \ldots, \alpha_r, \tag{8.12}$$

where δ is the primitive null root. If

$$\theta = \sum_{i=1}^{r} a_i \alpha_i, \tag{8.13}$$

then

$$\boxed{\delta = \sum_{i=0}^{r} a_i \alpha_i, \qquad a_0 = 1.} \tag{8.14}$$

The central element instead involves the comarks $a_i^\vee$,

$$\boxed{K = \sum_{i=0}^{r} a_i^\vee h_i.} \tag{8.15}$$

For simply laced algebras marks and comarks coincide; for nonsimply laced algebras they need not.

8.4 Affine $A_1^{(1)}$

The affine Cartan matrix is

$$A_1^{(1)} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}. \tag{8.16}$$

The null root and center are

$$\delta = \alpha_0 + \alpha_1, \qquad K = h_0 + h_1. \tag{8.17}$$

An integrable highest weight at level $k \in \mathbb{Z}_{\geq 0}$ has affine Dynkin labels (λ_0, λ_1) obeying

$$\lambda_0 \geq 0, \qquad \lambda_1 \geq 0, \qquad \boxed{\lambda_0 + \lambda_1 = k.} \tag{8.18}$$

Thus the allowed finite $SU(2)$ spins are

$$j = 0, \frac{1}{2}, 1, \ldots, \frac{k}{2}. \tag{8.19}$$

8.5 Affine Weyl group

The affine Weyl group is generated by reflections in the affine simple roots. It is the semidirect product of the finite Weyl group with translations by the coroot lattice (up to the standard convention for untwisted affine algebras):

$$W_{\text{aff}} \simeq W \ltimes Q^\vee. \tag{8.20}$$

Geometrically, finite Weyl chambers are replaced by alcoves bounded by the affine hyperplane associated with the highest root. Integrable highest weights at fixed level occupy the corresponding fundamental alcove.

8.6 Oscillator realizations

The notes also construct Lie generators from bilinears of harmonic-oscillator operators. If $a_i, a_i^\dagger$ satisfy

$$[a_i, a_j^\dagger] = \delta_{ij}, \tag{8.21}$$

then

$$E_{ij} = a_i^\dagger a_j \tag{8.22}$$

obey the $\mathfrak{gl}(N)$ algebra

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{il} E_{kj}. \tag{8.23}$$

The traceless part realizes $\mathfrak{su}(N)$ after imposing the appropriate Hermiticity convention. Such oscillator constructions are particularly useful for building highest-weight representations and affine currents in free-field realizations.

Chapter 9

Lorentz and Poincaré groups, one-particle representations, and light-front form

9.1 Lorentz algebra and its complexification

With mostly-plus metric $\eta = \mathrm{diag}(-,+,+,+)$, the Lorentz generators satisfy

$$[M_{\mu\nu}, M_{\rho\sigma}] = \mathrm{i}\left(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma} + \eta_{\nu\sigma}M_{\mu\rho}\right). \tag{9.1}$$

Define rotations and boosts by

$$J_i = \frac{1}{2}\epsilon_{ijk}M_{jk}, \qquad K_i = M_{0i}. \tag{9.2}$$

Then

$$[J_i, J_j] = \mathrm{i}\epsilon_{ijk}J_k, \tag{9.3}$$
$$[J_i, K_j] = \mathrm{i}\epsilon_{ijk}K_k, \tag{9.4}$$
$$[K_i, K_j] = -\mathrm{i}\epsilon_{ijk}J_k. \tag{9.5}$$

Introduce

$$A_i = \frac{1}{2}(J_i + \mathrm{i}K_i), \qquad B_i = \frac{1}{2}(J_i - \mathrm{i}K_i), \tag{9.6}$$

so that

$$[A_i, A_j] = \mathrm{i}\epsilon_{ijk}A_k, \qquad [B_i, B_j] = \mathrm{i}\epsilon_{ijk}B_k, \qquad [A_i, B_j] = 0. \tag{9.7}$$

Finite-dimensional Lorentz representations are therefore labelled by (j_L, j_R).

The standard finite-dimensional fields are

$$\text{left Weyl spinor}: (\tfrac{1}{2}, 0), \tag{9.8}$$
$$\text{right Weyl spinor}: (0, \tfrac{1}{2}), \tag{9.9}$$
$$\text{four-vector}: (\tfrac{1}{2}, \tfrac{1}{2}), \tag{9.10}$$
$$\text{self-dual two-form}: (1, 0), \tag{9.11}$$
$$\text{anti-self-dual two-form}: (0, 1). \tag{9.12}$$

9.2 Poincaré algebra and Casimirs

Adding translations P_μ gives

$$[P_\mu, P_\nu] = 0, \tag{9.13}$$

$$[M_{\mu\nu}, P_\rho] = \mathrm{i}(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu). \tag{9.14}$$

The two Casimirs in four dimensions are

$$P^2 = P_\mu P^\mu \tag{9.15}$$

and the square of the Pauli–Lubanski vector

$$W^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}P_\nu M_{\rho\sigma}, \qquad W^2 = W_\mu W^\mu. \tag{9.16}$$

For a massive irreducible representation, with our mostly-plus convention,

$$P^2 = -m^2, \qquad W^2 = m^2 s(s+1). \tag{9.17}$$

(The sign of the second equation changes if a different metric and epsilon convention is adopted.)

9.3 Wigner's little-group construction

Choose a standard momentum k. The subgroup of Lorentz transformations leaving k invariant is the little group. One-particle states are induced from unitary irreducible representations of this subgroup.

For $m > 0$, take $k = (m, 0, 0, 0)$; the little group is $SO(3)$, or $SU(2)$ on the spin cover. States are labelled by spin s and $m_s = -s, \ldots, s$.

For a massless standard momentum, the little group is

$$ISO(2) = SO(2) \ltimes \mathbb{R}^2. \tag{9.18}$$

If the two translation generators act trivially, one obtains the familiar helicity representations. If they act nontrivially, one obtains continuous-spin representations.

9.4 Momentum-space generators and Newton–Wigner form

For a massive particle of spin s, a standard momentum-space realization has

$$\mathbf{J} = -\mathrm{i}\,\mathbf{p} \times \boldsymbol{\nabla}_{\mathbf{p}} + \mathbf{S}, \tag{9.19}$$

and, up to convention-dependent Hermitian measure terms,

$$\mathbf{K} = \mathrm{i}p^0\boldsymbol{\nabla}_{\mathbf{p}} - \frac{\mathbf{S} \times \mathbf{p}}{p^0 + m}, \qquad p^0 = \sqrt{\mathbf{p}^2 + m^2}. \tag{9.20}$$

These expressions underlie the Newton–Wigner discussion in the notes. The exact form of the derivative term depends on whether states are normalized with d^3p, $d^3p/(2p^0)$, or an equivalent measure.

9.5 Light-front coordinates

Define

$$x^{\pm} = \frac{x^0 \pm x^3}{\sqrt{2}}, \qquad p^{\pm} = \frac{p^0 \pm p^3}{\sqrt{2}}. \tag{9.21}$$

The mass shell condition is

$$2p^+p^- - \mathbf{p}_\perp^2 = m^2, \tag{9.22}$$

therefore

$$\boxed{p^- = \frac{\mathbf{p}_\perp^2 + m^2}{2p^+}.} \tag{9.23}$$

Taking x^+ as light-front time separates the ten Poincaré generators into seven kinematical generators, which preserve the $x^+ = 0$ hypersurface, and three dynamical generators. This is the Dirac front form used in the manuscript.

Chapter 10

Gauge transformations, Weyl spinors, and Dirac fields

10.1 Local gauge symmetry

Let a field transform as

$$\psi(x) \longrightarrow U(x)\psi(x), \qquad U(x) \in G. \tag{10.1}$$

Choose

$$D_\mu = \partial_\mu + \mathrm{i}gA_\mu, \qquad A_\mu = A^a_\mu T^a. \tag{10.2}$$

Covariance,

$$D'_\mu \psi' = U D_\mu \psi, \tag{10.3}$$

requires

$$\boxed{A'_\mu = U A_\mu U^{-1} - \frac{\mathrm{i}}{g}(\partial_\mu U)U^{-1}.} \tag{10.4}$$

The field strength follows from

$$[D_\mu, D_\nu] = \mathrm{i}gF_{\mu\nu}, \tag{10.5}$$

so

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \mathrm{i}g[A_\mu, A_\nu], \tag{10.6}$$

and transforms covariantly,

$$F'_{\mu\nu} = U F_{\mu\nu} U^{-1}. \tag{10.7}$$

With anti-Hermitian generators or $D_\mu = \partial_\mu - \mathrm{i}gA_\mu$ all signs change consistently.

10.2 Two-component spinor conventions

Let $\alpha = 1, 2$ and $\dot{\alpha} = 1, 2$ denote the two Lorentz $SL(2, \mathbb{C})$ factors. For the mostly-plus metric used in this document one convenient convention is

$$\sigma^\mu_{\alpha\dot{\alpha}} = (\mathbf{1}, \vec{\sigma}), \qquad \bar{\sigma}^{\mu\,\dot{\alpha}\alpha} = (-\mathbf{1}, \vec{\sigma}), \tag{10.8}$$

which obeys

$$\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu = 2\eta^{\mu\nu}\mathbf{1}. \tag{10.9}$$

Many amplitude references instead use the mostly-minus metric and $\bar{\sigma}^\mu = (\mathbf{1}, -\vec{\sigma})$; translating between the two requires changing the corresponding signs consistently. Spinor indices are raised and lowered with

$$\epsilon^{12} = -\epsilon^{21} = 1, \qquad \psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta. \tag{10.10}$$

A left-handed Weyl spinor transforms as $(\frac{1}{2}, 0)$ and a right-handed one as $(0, \frac{1}{2})$. A four-component Dirac spinor is

$$\Psi = \begin{pmatrix} \chi_\alpha \\ \bar{\eta}^{\dot{\alpha}} \end{pmatrix}, \tag{10.11}$$

with kinetic term

$$\mathcal{L}_{\text{kin}} = \mathrm{i}\bar{\Psi}\gamma^\mu D_\mu \Psi. \tag{10.12}$$

A Dirac mass couples conjugate chiral representations,

$$\mathcal{L}_m = -m(\eta\chi + \bar{\chi}\bar{\eta}). \tag{10.13}$$

A Majorana mass is possible only when gauge quantum numbers permit an invariant bilinear of the required symmetry.

10.3 Chirality and helicity

Chirality labels Lorentz representations and is defined for fields independently of momentum. Helicity is the projection of angular momentum along momentum,

$$h = \frac{\mathbf{J} \cdot \mathbf{p}}{|\mathbf{p}|}. \tag{10.14}$$

For positive-energy massless particles, chirality and helicity are correlated; for massive particles they are distinct because boosts can reverse the direction of the three-momentum relative to spin.

Chapter 11

Representation theory in quantum field theory and gauge anomalies

11.1 Real, pseudoreal, and complex representations

Let $R(g)$ be a unitary representation. If there exists a nonsingular matrix S such that

$$R(g)^* = SR(g)S^{-1}, \tag{11.1}$$

then R is equivalent to its complex conjugate. It is

- real if S may be chosen symmetric;
- pseudoreal (quaternionic) if S may be chosen antisymmetric;
- complex if $R \not\simeq R^*$.

For the $SU(2)$ doublet one may take $S = \mathrm{i}\sigma_2$, so the doublet is pseudoreal. Adjoint representations of compact groups are real.

Every $R \otimes \bar{R}$ contains a singlet, corresponding to the invariant identity map. Moreover, if R is a nontrivial representation of a *simple Lie algebra*, the generator map $\mathfrak{g} \to \mathrm{End}(R) \simeq R \otimes R^*$ has zero kernel (the kernel would be an ideal), so an adjoint copy also occurs. For the $SU(N)$ fundamental this familiar statement is exactly

$$\mathbf{N} \otimes \bar{\mathbf{N}} = \mathbf{1} \oplus \mathrm{Adj}. \tag{11.2}$$

For the vector of $SO(N)$,

$$\mathbf{N} \otimes \mathbf{N} = \mathbf{1} \oplus \Lambda^2\mathbf{N} \oplus \mathrm{Sym}^2_0\,\mathbf{N}, \tag{11.3}$$

where $\Lambda^2\mathbf{N}$ is the adjoint.

11.2 Quadratic indices for products

For a product group $G_1 \times G_2$, a field in (R_1, R_2) contributes to the G_1 quadratic index with multiplicity $\dim R_2$:

$$T_{G_1}(R_1, R_2) = \dim(R_2) T_{G_1}(R_1). \tag{11.4}$$

This elementary fact is repeatedly used in beta functions and anomaly sums.

11.3 Perturbative chiral anomaly

For left-handed Weyl fermions, the local gauge anomaly is proportional to

$$d_R^{abc} = \mathrm{Tr}_R\Big(T^a\{T^b, T^c\}\Big)\,. \tag{11.5}$$

The theory is perturbatively anomaly free when

$$\sum_{\text{LH Weyl } f} d_{R_f}^{abc} = 0. \tag{11.6}$$

Conjugate representations contribute with opposite sign. Real and pseudoreal representations have vanishing cubic perturbative anomaly.

Chapter 12

Color decomposition of non-Abelian gauge amplitudes

12.1 Color factors and completeness

For $SU(N)$ in the fundamental normalization

$$\mathrm{Tr}(T^aT^b) = \frac{1}{2}\delta^{ab}, \tag{12.1}$$

we have

$$[T^a, T^b] = \mathrm{i} f^{abc} T^c, \tag{12.2}$$

$$f^{abc} = -2\mathrm{i}\,\mathrm{Tr}([T^a, T^b]T^c), \tag{12.3}$$

$$(T^a)_i{}^j (T^a)_k{}^l = \frac{1}{2}\left(\delta_i{}^l \delta_k{}^j - \frac{1}{N}\delta_i{}^j \delta_k{}^l\right). \tag{12.4}$$

The handwritten notes sometimes use $\mathrm{Tr}(T^aT^b) = \delta^{ab}$. All color identities then acquire the corresponding overall rescaling. Here we use the standard high-energy convention above.

12.2 Tree-level trace decomposition

The color-dressed n-gluon tree amplitude can be decomposed as

$$\boxed{\mathcal{A}_n^{\text{tree}} = g^{n-2} \sum_{\sigma\in S_n/Z_n} \mathrm{Tr}(T^{a_{\sigma(1)}} \cdots T^{a_{\sigma(n)}}) A_n(\sigma(1), \dots, \sigma(n)).} \tag{12.5}$$

The partial amplitudes A_n are gauge invariant and contain the kinematic singularities; color is isolated in the trace basis. Cyclicity gives

$$A_n(1, 2, \dots, n) = A_n(2, 3, \dots, n, 1). \tag{12.6}$$

Reflection gives

$$\boxed{A_n(1, 2, \dots, n) = (-1)^n A_n(n, n-1, \dots, 1).} \tag{12.7}$$

12.3 $U(1)$ decoupling identity

Replacing one generator by the identity corresponds to an abelian gauge boson, which has no non-Abelian self-interaction. Therefore

$$\boxed{\sum_{k=1}^{n-1} A_n(1,2,\ldots,k,n,k+1,\ldots,n-1)=0.} \tag{12.8}$$

This is the $U(1)$-decoupling relation displayed graphically in the notes.

12.4 Color matrices in squared amplitudes

If a chosen color basis is $\{C_I\}$ and

$$\mathcal{A}=\sum_I C_I A_I, \tag{12.9}$$

then the color-summed square has the form

$$\sum_{\text{colors}} |\mathcal{A}|^2=\sum_{I,J} A_I^* \mathcal{C}_{IJ}\, A_J, \qquad \mathcal{C}_{IJ}=\sum_{\text{colors}} C_I^* C_J. \tag{12.10}$$

Thus all group theory resides in a finite color metric $\mathcal{C}_{IJ}$, while kinematics resides in the vector of partial amplitudes.

Chapter 13

Spinor-helicity variables and gauge-boson polarizations

13.1 Null momenta as bispinors

For a null four-momentum,

$$p^2 = 0, \tag{13.1}$$

the 2×2 bispinor matrix has rank one,

$$p_{\alpha\dot{\alpha}} = p_\mu \sigma^\mu_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}. \tag{13.2}$$

We write

$$|p\rangle_\alpha = \lambda_\alpha, \qquad |p]_{\dot{\alpha}} = \tilde{\lambda}_{\dot{\alpha}}. \tag{13.3}$$

The Lorentz-invariant spinor products are

$$\langle ij\rangle = \epsilon^{\alpha\beta} \lambda_{i\alpha} \lambda_{j\beta}, \tag{13.4}$$

$$[ij] = \epsilon^{\dot{\alpha}\dot{\beta}} \tilde{\lambda}_{i\dot{\alpha}} \tilde{\lambda}_{j\dot{\beta}}. \tag{13.5}$$

They are antisymmetric,

$$\langle ij\rangle = -\langle ji\rangle, \qquad [ij] = -[ji]. \tag{13.6}$$

With a standard mostly-minus amplitude convention,

$$\boxed{\langle ij\rangle[ji] = 2p_i \cdot p_j.} \tag{13.7}$$

The sign should be adjusted consistently if the metric convention is changed.

13.2 Schouten identity

Because the undotted spinor space is two-dimensional,

$$\boxed{\langle ij\rangle\langle kl\rangle + \langle jk\rangle\langle il\rangle + \langle ki\rangle\langle jl\rangle = 0,} \tag{13.8}$$

and an analogous identity holds for square brackets. This identity accounts for many of the algebraic simplifications carried out in the manuscript.

13.3 Little-group scaling

The factorization of a null momentum is unchanged under

$$\lambda_i \to t_i \lambda_i, \qquad \tilde{\lambda}_i \to t_i^{-1} \tilde{\lambda}_i. \tag{13.9}$$

For a particle of helicity h_i, an amplitude scales as

$$\boxed{\mathcal{A}_n \to t_i^{-2h_i} \mathcal{A}_n.} \tag{13.10}$$

Thus little-group covariance provides a powerful check on any proposed helicity amplitude.

13.4 Polarization vectors

For a massless vector of momentum p and an arbitrary null reference momentum q,

$$\varepsilon_+^\mu(p;q) = \frac{\langle q|\gamma^\mu|p]}{\sqrt{2}\langle qp\rangle}, \tag{13.11}$$

$$\varepsilon_-^\mu(p;q) = \frac{\langle p|\gamma^\mu|q]}{\sqrt{2}[pq]}. \tag{13.12}$$

They obey

$$p \cdot \varepsilon_\pm = 0. \tag{13.13}$$

Changing q shifts the polarization by a multiple of p^μ,

$$\varepsilon^\mu(p;q') - \varepsilon^\mu(p;q) \propto p^\mu, \tag{13.14}$$

which is precisely a gauge transformation. This proves the reference-spinor independence of gauge-invariant amplitudes.

13.5 Tree-level helicity selection rules

For $n \geq 4$, pure Yang–Mills tree amplitudes satisfy

$$A_n(+,+,\ldots,+) = 0, \tag{13.15}$$

and

$$A_n(-,+,\ldots,+) = 0 \tag{13.16}$$

(up to cyclic relabeling of the single negative-helicity leg). Three-point amplitudes require a separate statement: with real Minkowski momenta they vanish kinematically, while for complex momenta the $--+$ and $++-$ amplitudes are nonzero. The first nonzero family is the MHV sector with two negative helicities. The Parke–Taylor formula, which is a natural completion of the notes, is

$$\boxed{A_n^{\text{tree}}(1^+,\ldots,i^-,\ldots,j^-,\ldots,n^+) = \mathrm{i}\,\frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}} \tag{13.17}$$

for the standard color-ordered normalization (overall phase convention may vary). It has exactly the required little-group weights and factorization properties.

Chapter 14

Conformal algebra and characters in general dimension

14.1 Conformal group and real forms

In Lorentzian signature the conformal algebra in d dimensions is

$$\mathfrak{so}(d, 2), \tag{14.1}$$

whereas in Euclidean signature it is

$$\mathfrak{so}(d + 1, 1). \tag{14.2}$$

The complexified algebra is $\mathfrak{so}(d{+}2, \mathbb{C})$ in either case. The generators are Lorentz transformations $M_{\mu\nu}$, translations P_μ, special conformal transformations K_μ, and dilatations D. A consistent convention is

$$[M_{\mu\nu}, P_\rho] = \mathrm{i}(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu), \tag{14.3}$$
$$[M_{\mu\nu}, K_\rho] = \mathrm{i}(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu), \tag{14.4}$$
$$[D, P_\mu] = \mathrm{i}P_\mu, \tag{14.5}$$
$$[D, K_\mu] = -\mathrm{i}K_\mu, \tag{14.6}$$
$$[P_\mu, K_\nu] = 2\mathrm{i}(\eta_{\mu\nu}D - M_{\mu\nu}), \tag{14.7}$$
$$[P_\mu, P_\nu] = [K_\mu, K_\nu] = 0. \tag{14.8}$$

Signs vary with Hermiticity conventions; the algebraic content is invariant.

14.2 Compact energy basis

For positive-energy representation theory it is convenient to pass to the compact basis in which the maximal compact subalgebra is

$$\mathfrak{so}(2) \oplus \mathfrak{so}(d). \tag{14.9}$$

The $SO(2)$ generator H is the conformal Hamiltonian, whose eigenvalue is the scaling dimension Δ. Raising operators increase Δ by one and transform as the vector of $SO(d)$; their Hermitian conjugates lower it. A conformal primary is a lowest-energy state

$$|\Delta, \ell\rangle, \tag{14.10}$$

annihilated by all lowering operators. Acting with translations generates the conformal Verma module of descendants.

14.3 Orthonormal-basis realization of $\mathfrak{so}(d+2)$

The manuscript develops explicit M_{AB} generators satisfying

$$[M_{AB}, M_{CD}] = \mathrm{i}(\eta_{AC}M_{BD} - \eta_{AD}M_{BC} - \eta_{BC}M_{AD} + \eta_{BD}M_{AC}). \tag{14.11}$$

This basis is useful for deriving Cartan generators and root operators for $SO(d+2)$ in both even and odd dimensions. For $d = 2r$, the compact rotation algebra is $D_r = \mathfrak{so}(2r)$; for $d = 2r+1$ it is $B_r = \mathfrak{so}(2r+1)$.

14.4 Rotation highest weights

For $SO(2r)$ one may label a highest weight by orthonormal coordinates

$$\ell = (\ell_1, \dots, \ell_r), \qquad \ell_1 \geq \ell_2 \geq \cdots \geq |\ell_r|. \tag{14.12}$$

For $SO(2r+1)$,

$$\ell_1 \geq \ell_2 \geq \cdots \geq \ell_r \geq 0. \tag{14.13}$$

The ℓ_i are all integers or all half-integers, according to tensorial or spinorial representations.

For $SO(2r)$, with simple roots

$$\alpha_i = e_i - e_{i+1} \quad (i = 1, \dots, r-1), \qquad \alpha_r = e_{r-1} + e_r, \tag{14.14}$$

the Dynkin labels are

$$a_i = \ell_i - \ell_{i+1}, \qquad i = 1, \dots, r-2, \tag{14.15}$$
$$a_{r-1} = \ell_{r-1} - \ell_r, \tag{14.16}$$
$$a_r = \ell_{r-1} + \ell_r. \tag{14.17}$$

For $SO(2r+1)$,

$$a_i = \ell_i - \ell_{i+1}, \qquad i = 1, \dots, r-1, \tag{14.18}$$
$$a_r = 2\ell_r. \tag{14.19}$$

14.5 Characters of $SO(2r)$

Let $x_i = e^{\mathrm{i}\theta_i}$ parametrize a maximal torus and define

$$k_j = \ell_j + r - j. \tag{14.20}$$

The Weyl character formula may be written as

$$\boxed{\chi_\ell^{D_r}(x) = \frac{\det(x_i^{k_j} + x_i^{-k_j}) + \det(x_i^{k_j} - x_i^{-k_j})}{\det(x_i^{r-j} + x_i^{-(r-j)})}.} \tag{14.21}$$

The determinants are $r \times r$. If $\ell_r = 0$, the second determinant in the numerator vanishes because its final column is zero. When $\ell_r \neq 0$, the two terms distinguish the chiral data of D_r.

The dimension formula is

$$\boxed{\dim_{D_r}(\ell) = \prod_{1 \leq i < j \leq r} \frac{(\ell_i - \ell_j + j - i)(\ell_i + \ell_j + 2r - i - j)}{(j-i)(2r - i - j)}.} \tag{14.22}$$

14.6 Characters of $SO(2r+1)$

Define

$$k_j = \ell_j + r - j + \frac{1}{2}. \tag{14.23}$$

Then

$$\boxed{\chi_\ell^{B_r}(x) = \frac{\det(x_i^{k_j} - x_i^{-k_j})}{\det(x_i^{r-j+1/2} - x_i^{-(r-j+1/2)})}.} \tag{14.24}$$

The Weyl dimension formula becomes

$$\begin{aligned}\dim_{B_r}(\ell) = &\prod_{i=1}^{r} \frac{2\ell_i + 2r - 2i + 1}{2r - 2i + 1} \\ &\times \prod_{1 \le i < j \le r} \frac{(\ell_i - \ell_j + j - i)(\ell_i + \ell_j + 2r - i - j + 1)}{(j-i)(2r - i - j + 1)}.\end{aligned} \tag{14.25}$$

14.7 Descendant generating functions

Each translation raises the dimension by one and transforms as the vector of $SO(d)$. The symmetric algebra of translations therefore gives the universal factor

$$P^{(2r)}(s,x) = \prod_{i=1}^{r} \frac{1}{(1 - sx_i)(1 - sx_i^{-1})} \tag{14.26}$$

for even $d = 2r$, and

$$P^{(2r+1)}(s,x) = \frac{1}{1-s} \prod_{i=1}^{r} \frac{1}{(1 - sx_i)(1 - sx_i^{-1})} \tag{14.27}$$

for odd $d = 2r+1$.

For a generic (long) conformal multiplet,

$$\boxed{\chi_{\Delta,\ell}^{\text{long}}(s,x) = s^\Delta \chi_\ell^{SO(d)}(x) P^{(d)}(s,x).} \tag{14.28}$$

This is the clean generating-function form of the Verma-module characters derived at the end of the manuscript.

14.8 Unitarity bounds and shortening

Positive-energy unitarity requires lower bounds on Δ. Important standard cases are

$$\text{scalar:} \qquad \Delta \ge \frac{d-2}{2}, \tag{14.29}$$

$$\text{spinor:} \qquad \Delta \ge \frac{d-1}{2}, \tag{14.30}$$

$$\text{symmetric traceless spin } \ell \ge 1: \qquad \Delta \ge \ell + d - 2. \tag{14.31}$$

At saturation, descendants become null and must be divided out. For a general mixed-symmetry $SO(d)$ highest weight, the bound depends on the position of the first drop among the ordered ℓ_i; this is the origin of the several branches worked out in the source pages 209–233.

Verification note. There is no single formula $\Delta \geq \ell_1 + d - 2$ that covers every spinorial and mixed-symmetry representation. The source correctly moves to special branches later, but the earlier shorthand must be read with this caveat.

14.9 Free scalar and conserved-current characters

For a scalar saturating

$$\Delta_\phi = \frac{d-2}{2}, \tag{14.32}$$

the null descendant is the equation of motion $P^2|\phi\rangle$. Hence

$$\boxed{\chi_\phi(s,x) = s^{(d-2)/2}(1-s^2)P^{(d)}(s,x).} \tag{14.33}$$

For a symmetric traceless conserved current of spin $\ell \geq 1$ at

$$\Delta = \ell + d - 2, \tag{14.34}$$

the divergence is null, yielding

$$\boxed{\chi_{J_\ell}(s,x) = s^{\ell+d-2}\left[\chi_\ell^{SO(d)}(x) - s\chi_{\ell-1}^{SO(d)}(x)\right]P^{(d)}(s,x).} \tag{14.35}$$

For the stress tensor, $\ell = 2$ and $\Delta = d$.

14.10 Free spinor character

A free spinor has

$$\Delta_\psi = \frac{d-1}{2}. \tag{14.36}$$

The Dirac equation removes the gamma-trace descendant at level one. In even $d = 2r$, for a chiral spinor $S_\pm$, translation produces $V \otimes S_\pm$ and the Dirac null module transforms in the opposite chirality $S_\mp$. Thus, in compact notation,

$$\boxed{\chi_{\psi_\pm}(s,x) = s^{(d-1)/2}\left[\chi_{S_\pm}(x) - s\chi_{S_\mp}(x)\right]P^{(d)}(s,x),} \tag{14.37}$$

with the obvious single-spinor analogue in odd dimensions. This matches the subtraction logic developed in the handwritten examples.

14.11 Even versus odd dimensions

For $d = 2r$, the compact rotation algebra is D_r and there are two chiral spinor nodes; the Weyl group permits sign flips of an even number of orthonormal weight coordinates. For $d = 2r+1$, the compact rotation algebra is B_r and arbitrary independent sign flips occur. This difference explains the extra reflection cases treated on the final pages of the manuscript and why the odd-dimensional character formula has an additional $(1-s)^{-1}$ descendant factor.

14.12 Summary of the character construction

The full logic of the final part of the notes can be condensed into four steps:

1. Choose the $SO(d)$ highest weight ℓ and scaling dimension Δ.
2. Form the Verma character $s^{\Delta}\chi_{\ell}P^{(d)}$.
3. Determine whether Δ lies above, at, or below a unitarity threshold.
4. At a shortening point, subtract the character(s) of singular submodules, including their intersections according to the shifted Weyl-group structure.

This is the representation-theoretic meaning of the alternating combinations of Verma-module characters written on source pages 209–233.

Appendix A

Conventions and quick-reference formulae

A.1 Cartan and root conventions

$$A_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_i, \alpha_i)}, \qquad [h_i, e_j] = A_{ij} e_j, \qquad [h_i, f_j] = -A_{ij} f_j. \tag{A.1}$$

Some texts use the transpose convention $2(\alpha_i, \alpha_j)/(\alpha_j, \alpha_j)$; formulas must then be transposed consistently.

A.2 Central representation-theory formulae

$$C_{2,\mathrm{can}}(\lambda) = (\lambda, \lambda + 2\rho), \tag{A.2}$$

$$\dim L(\lambda) = \prod_{\alpha > 0} \frac{(\lambda + \rho, \alpha)}{(\rho, \alpha)}, \tag{A.3}$$

$$\chi_\lambda = \frac{\sum_{w \in W} (-1)^{\ell(w)} e^{w(\lambda+\rho)}}{\sum_{w \in W} (-1)^{\ell(w)} e^{w(\rho)}}, \tag{A.4}$$

$$T(R) \dim \mathfrak{g} = C_2(R) \dim R. \tag{A.5}$$

A.3 Gauge and amplitude formulae

$$[D_\mu, D_\nu] = \mathrm{i} g F_{\mu\nu}, \tag{A.6}$$

$$(T^a)_i{}^j (T^a)_k{}^l = \frac{1}{2}\left(\delta_i{}^l \delta_k{}^j - \frac{1}{N} \delta_i{}^j \delta_k{}^l\right), \tag{A.7}$$

$$p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}, \tag{A.8}$$

$$\langle ij \rangle [ji] = 2 p_i \cdot p_j. \tag{A.9}$$

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