Effective Field Theory

Invariant Operators, Hilbert Series, Conformal Representations, and Heat-Kernel Methods

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Contents

Chapter 1

Local invariant operators and dimensional analysis

1.1 Natural units and canonical dimensions

With $\hbar = c = 1$, mass, inverse length, and inverse time have the same dimension. In d spacetime dimensions,

$$S = \int \mathrm{d}^d x \, \mathcal{L}, \qquad [S] = 0, \qquad [\mathrm{d}^d x] = -d, \qquad [\mathcal{L}] = d.$$

In four dimensions,

$$[\phi] = 1, \qquad [A_\mu] = 1, \qquad [\psi] = \frac{3}{2}, \qquad [\mathcal{D}_\mu] = 1, \qquad [F_{\mu\nu}] = 2.$$

For a local operator $\mathcal{O}_i$ of canonical dimension d_i,

$$\mathcal{L}_{\mathrm{EFT}} = \sum_i C_i \mathcal{O}_i, \qquad [C_i] = 4 - d_i.$$

It is often useful to expose the heavy scale Λ explicitly:

$$\mathcal{L}_{\mathrm{EFT}} = \mathcal{L}_{\mathrm{ren}} + \sum_{d_i > 4} \frac{c_i}{\Lambda^{d_i - 4}} \mathcal{O}_i,$$

where c_i is dimensionless in four dimensions.

1.2 The building blocks

For local gauge-invariant polynomial operators, one works with matter fields, field strengths, and covariant derivatives. The gauge connection A_μ is not by itself gauge invariant; its gauge-covariant information is packaged into

$$\mathcal{D}_\mu = \partial_\mu - \mathrm{i} g A^a_\mu T^a, \qquad [\mathcal{D}_\mu, \mathcal{D}_\nu] = -\mathrm{i} g F^a_{\mu\nu} T^a.$$

The statement that “the gauge field does not appear” must therefore be read as a statement about *local gauge-invariant bases*; nonlocal Wilson lines are a separate class of observables.

1.3 Symmetry restricts the operator basis

Operators must be singlets under the spacetime and internal symmetries being imposed. For a real scalar with no additional symmetry, the renormalizable four-dimensional Lagrangian may contain

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \left(a_1\phi + \frac{1}{2}m^2\phi^2 + \frac{a_3}{3!}\phi^3 + \frac{\lambda}{4!}\phi^4\right).$$

A $\mathbb{Z}_2$ symmetry $\phi \mapsto -\phi$ removes odd powers. For a complex scalar with a global $U(1)$ symmetry, $\phi \mapsto \mathrm{e}^{\mathrm{i}\theta}\phi$, the potential is a function of $\phi^\dagger\phi$:

$$V = m^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2 + \cdots .$$

Representative invariant structures are summarized in table 1.1.

Table 1.1: Basic invariant operator structures in four dimensions.

Category	schematic field content	representative operator
Scalar potential	ϕ^n	$(\phi^\dagger\phi)^n$
Scalar kinetic	$\phi^2 D^2$	$(D_\mu\phi)^\dagger D^\mu\phi$
Fermion kinetic	$\psi^2 D$	$\mathrm{i}\bar{\psi}\gamma^\mu D_\mu\psi$
Gauge kinetic	X^2	$-\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}$
Yukawa	$\psi^2\phi$	$\bar{\psi}_i\phi\psi_j$

1.4 Total derivatives and IBP equivalence

Two Lagrangian densities that differ by a total derivative give the same action when the corresponding boundary term vanishes:

$$\mathcal{L}' = \mathcal{L} + \partial_\mu J^\mu, \qquad S' - S = \int_{\partial\mathcal{M}} \mathrm{d}\Sigma_\mu\, J^\mu.$$

A manifold without boundary is sufficient, but not necessary; appropriate falloff or boundary conditions are equally adequate.

Chapter 2

Hilbert series and plethystic exponentials

2.1 Generating invariants

For a single complex scalar with $U(1)$ charge $+1$, the only polynomial invariants are powers of $\phi^\dagger\phi$. Introducing a grading variable t,

$$H(t) = \sum_{n=0}^{\infty} t^{2n} = \frac{1}{1-t^2}.$$

The grading variable should be distinguished from the field itself; this makes the Hilbert series a genuine generating function for operator degree.

For two real singlet scalars ϕ_1, ϕ_2, the degree-two monomials are

$$\phi_1^2, \qquad \phi_1\phi_2, \qquad \phi_2^2,$$

so the multiplicity at degree two is three.

2.2 Molien–Weyl projection

The singlet part of a representation is projected by Haar integration. For the $U(1)$ example,

$$H(t) = \oint_{|z|=1} \frac{\mathrm{d}z}{2\pi \mathrm{i} z} \frac{1}{(1-tz)(1-tz^{-1})} = \frac{1}{1-t^2}.$$

More generally, if fields transform in representations R_i of a compact group G, the Hilbert series is a group integral of a plethystic generating function,

$$H(\{t_i\}) = \int_G \mathrm{d}\mu_G \ \mathrm{PE}\left[\sum_i t_i \chi_{R_i}\right],$$

with a fermionic modification for Grassmann fields.

2.3 Plethystic exponentials

For a bosonic single-particle character $f(x)$,

$$\mathrm{PE}_B[f](x) = \exp\left[\sum_{r=1}^{\infty} \frac{1}{r} f(x^r)\right],$$

which builds symmetric tensor powers. For fermions,

$$\mathrm{PE}_F[f](x) = \exp\left[\sum_{r=1}^{\infty} \frac{(-1)^{r+1}}{r} f(x^r)\right],$$

which builds antisymmetric tensor powers. If a field ϕ in representation R is assigned fugacity t, then

$$\mathrm{PE}_B[t\chi_R(z)] = \exp\left[\sum_{r\geq 1} \frac{t^r}{r} \chi_R(z^r)\right].$$

Characters multiply under tensor products:

$$\chi_{R\otimes R'}(g) = \chi_R(g)\chi_{R'}(g),$$

and compact-group character orthogonality,

$$\int_G \mathrm{d}\mu(g)\, \chi_R(g)^* \chi_{R'}(g) = \delta_{RR'},$$

explains why Haar integration extracts singlets.

Chapter 3

Compact Lie groups, maximal tori, and $SU(N)$ characters

3.1 Maximal tori and rank

A torus in a compact Lie group is a compact, connected, abelian subgroup. A maximal torus T is maximal under inclusion, and

$$\operatorname{rank} G = \dim T.$$

For $U(N)$,

$$T_{U(N)} = \operatorname{diag}(\mathrm{e}^{\mathrm{i}\theta_1}, \ldots, \mathrm{e}^{\mathrm{i}\theta_N}),$$

whereas for $SU(N)$ the determinant-one constraint leaves $N-1$ independent phases.

3.2 A useful $SU(N)$ torus parametrization

Introduce $z_i \in U(1)$ for $i = 1, \ldots, N-1$ and define the defining-representation eigenvalues

$$\epsilon_1 = z_1, \tag{3.1}$$

$$\epsilon_k = z_{k-1}^{-1} z_k, \qquad 2 \le k \le N-1, \tag{3.2}$$

$$\epsilon_N = z_{N-1}^{-1}. \tag{3.3}$$

Then $\prod_{a=1}^{N} \epsilon_a = 1$ automatically.

3.3 Dynkin labels, partitions, and Young diagrams

For $SU(N)$ an irrep is labelled by nonnegative Dynkin labels

$$(a_1, a_2, \ldots, a_{N-1}).$$

Equivalently, define row lengths

$$\lambda_k = \sum_{i=k}^{N-1} a_i, \qquad k = 1, \ldots, N-1, \qquad \lambda_N = 0.$$

The Young diagram has λ_k boxes in row k. A full column of N boxes is trivial in $SU(N)$, so one conventionally works with at most $N-1$ nonzero rows.

3.4 Weyl character formula for $SU(N)$

For partition $\lambda = (\lambda_1, \ldots, \lambda_N)$,

$$\boxed{\chi_\lambda(\epsilon) = \frac{\det\left(\epsilon_i^{\lambda_j+N-j}\right)_{i,j=1}^N}{\det\left(\epsilon_i^{N-j}\right)_{i,j=1}^N}}$$

with Vandermonde denominator

$$\Delta(\epsilon) = \prod_{1\le i<j\le N} (\epsilon_i - \epsilon_j).$$

3.4.1 $SU(2)$

Take $\epsilon = (z, z^{-1})$. The first few characters are

$$\chi_{\mathbf{1}} = 1, \qquad \chi_{\mathbf{2}} = z + z^{-1}, \qquad \chi_{\mathbf{3}} = z^2 + 1 + z^{-2},$$

and the tensor-product check is

$$\chi_{\mathbf{2}}^2 = \chi_{\mathbf{3}} + \chi_{\mathbf{1}}, \quad \Longleftrightarrow \quad \mathbf{2}\otimes\mathbf{2} = \mathbf{3}\oplus\mathbf{1}.$$

3.4.2 $SU(3)$

With $\epsilon = (z_1, z_1^{-1}z_2, z_2^{-1})$,

$$\chi_{\mathbf{3}} = z_1 + \frac{z_2}{z_1} + \frac{1}{z_2}, \qquad \chi_{\overline{\mathbf{3}}} = \chi_{\mathbf{3}}(z_1^{-1}, z_2^{-1}),$$

and

$$\chi_{\mathbf{8}} = \chi_{\mathbf{3}}\chi_{\overline{\mathbf{3}}} - 1, \quad \Longleftrightarrow \quad \mathbf{3}\otimes\overline{\mathbf{3}} = \mathbf{8}\oplus\mathbf{1}.$$

3.5 Weyl integration formula and Haar measure

For a class function f on a compact connected group,

$$\int_G \mathrm{d}\mu_G f = \frac{1}{|W|}\oint_T \mathrm{d}\mu_T \prod_{\alpha>0}(1-z^\alpha)(1-z^{-\alpha})f(z),$$

where W is the Weyl group. In particular,

$$\int_{SU(N)} \mathrm{d}\mu\, f = \frac{1}{N!}\prod_{k=1}^{N-1}\oint_{|z_k|=1}\frac{\mathrm{d}z_k}{2\pi\mathrm{i}z_k}\,\Delta(\epsilon)\Delta(\epsilon^{-1})f(z).$$

For $SU(2)$ this becomes

$$\int \mathrm{d}\mu_{SU(2)} f(z) = \frac{1}{2}\oint\frac{\mathrm{d}z}{2\pi\mathrm{i}z}(1-z^2)(1-z^{-2})f(z).$$

Chapter 4

Lorentz and conformal representations

4.1 The noncompactness issue

The four-dimensional Lorentz group $SO^+(3,1)$ is noncompact, so nontrivial finite-dimensional representations are not unitary. Nevertheless, local fields transform in finite-dimensional Lorentz representations, conveniently labelled after complexification by two spins (j_L, j_R).

The conformal real form depends on signature:

$$\text{Lorentzian } (3,1): \quad SO(4,2), \qquad \text{Euclidean 4d}: \quad SO(5,1).$$

In general, the conformal group of signature (p,q) is locally $SO(p+1,q+1)$.

4.2 Four-dimensional Lorentz characters

For $SU(2)$,

$$\chi_j(z) = z^{2j} + z^{2j-2} + \cdots + z^{-2j}.$$

Hence

$$\chi_{(j_L,j_R)}(\alpha,\beta) = \chi_{j_L}(\alpha)\chi_{j_R}(\beta).$$

The vector transforms as $(\frac{1}{2},\frac{1}{2})$. The two chiral parts of an antisymmetric field strength transform as

$$F_L \sim (1,0), \qquad F_R \sim (0,1),$$

with $F_{L,R} = \frac{1}{2}(F \pm \mathrm{i}\widetilde{F})$ in a common Minkowski convention.

4.3 Conformal descendants in four dimensions

The derivative-generating denominator is

$$P(D;\alpha,\beta) = \frac{1}{(1-D\alpha\beta)(1-D\alpha/\beta)(1-D\beta/\alpha)(1-D/(\alpha\beta))}.$$

A generic long conformal multiplet with primary $[\Delta;(j_L,j_R)]$ has character

$$\chi_{\text{long}} = D^{\Delta}\chi_{j_L}(\alpha)\chi_{j_R}(\beta)P(D;\alpha,\beta).$$

Shortening subtracts null descendants. For free fields,

$$\chi_\phi = D(1-D^2)P, \tag{4.1}$$

$$\chi_{\psi_L} = D^{3/2}P\left[\chi_{1/2}(\alpha) - D\chi_{1/2}(\beta)\right], \tag{4.2}$$

$$\chi_{\psi_R} = D^{3/2}P\left[\chi_{1/2}(\beta) - D\chi_{1/2}(\alpha)\right], \tag{4.3}$$

$$\chi_{F_L} = D^2P\left[\chi_1(\alpha) - D\chi_{1/2}(\alpha)\chi_{1/2}(\beta) + D^2\right], \tag{4.4}$$

$$\chi_{F_R} = \chi_{F_L}\big|_{\alpha\leftrightarrow\beta}. \tag{4.5}$$

The subtractions implement equations of motion and, for field strengths, Bianchi/conservation-type null relations.

For completeness, the generic four-dimensional short characters underlying those examples are

$$\chi_{[j_L+j_R+2;(j_L,j_R)]} = D^{j_L+j_R+2}P\left[\chi_{j_L}(\alpha)\chi_{j_R}(\beta) - D\chi_{j_L-1/2}(\alpha)\chi_{j_R-1/2}(\beta)\right], \qquad j_Lj_R \neq 0, \tag{4.6}$$

$$\chi_{[j+1;(j,0)]} = D^{j+1}P\left[\chi_j(\alpha) - D\chi_{j-1/2}(\alpha)\chi_{1/2}(\beta) + D^2\chi_{j-1}(\alpha)\right], \tag{4.7}$$

with the $(0,j)$ formula obtained by $\alpha \leftrightarrow \beta$. Terms with a negative spin label are absent. The free scalar is the special isolated case $D(1-D^2)P$.

4.4 Even-dimensional $SO(2r)$ characters

For $D_r \equiv SO(2r)$, a robust form of the Weyl character formula is

$$\chi_\lambda(z) = \frac{\sum_{w\in W(D_r)}(-1)^{\ell(w)}z^{w(\lambda+\rho)}}{\sum_{w\in W(D_r)}(-1)^{\ell(w)}z^{w(\rho)}}.$$

In the standard node convention, if $\lambda = (\ell_1,\dots,\ell_r)$ then

$$a_i = \ell_i - \ell_{i+1}, \qquad 1 \le i \le r-2, \tag{4.8}$$

$$a_{r-1} = \ell_{r-1} - \ell_r, \tag{4.9}$$

$$a_r = \ell_{r-1} + \ell_r. \tag{4.10}$$

The root system is $\pm e_i \pm e_j$, and the Weyl group has order

$$|W(D_r)| = 2^{r-1}r!.$$

The Haar factor may therefore be written as the product over positive roots,

$$\prod_{i<j}(1-z_iz_j)(1-z_i/z_j)(1-z_j/z_i)(1-z_i^{-1}z_j^{-1}).$$

For an even-dimensional vector descendant tower,

$$P^{(2r)}(q,z) = \prod_{i=1}^{r}\frac{1}{(1-qz_i)(1-q/z_i)}.$$

Chapter 5

IBP, EOM, conformal modules, and EFT counting

Source-note pages: 47–88.

5.1 Three distinct equivalence relations

It is essential to distinguish:

(i) algebraic identities (group identities, Fierz identities, Bianchi identities),

(ii) integration-by-parts equivalence in the action,

(iii) EOM redundancy generated by field redefinitions.

For IBP,

$$\mathcal{O}_1 - \mathcal{O}_2 = \partial_\mu J^\mu \quad \Rightarrow \quad \int \mathrm{d}^d x\, \mathcal{O}_1 \simeq \int \mathrm{d}^d x\, \mathcal{O}_2$$

when the boundary term vanishes. For EOM redundancy, an operator proportional to the lower-order EOM can be removed order by order by a local field redefinition. It is not literally zero off shell.

5.2 Derivative partitions and why $d > 1$ is harder

In a one-dimensional or purely schematic scalar problem, the difference of partition numbers can count structures modulo a total derivative. In more than one spacetime dimension, derivatives carry Lorentz indices, different tensor contractions exist, and commutators generate field strengths. The simple integer-partition subtraction therefore does not by itself solve the full EFT operator-counting problem.

A useful conceptual description of IBP is cohomological: local operators form a complex under the exterior derivative, and top forms modulo exact forms capture action densities modulo total derivatives. This viewpoint explains the exceptional terms that appear in conformal-character Hilbert-series formulae.

5.3 Conformal algebra and highest-weight modules

In d dimensions the conformal algebra is generated by translations P_μ, rotations/boosts $M_{\mu\nu}$, the dilatation D, and special conformal generators K_μ. With Hermitian generators in a common Lorentzian convention,

$$[D, P_\mu] = \mathrm{i}P_\mu, \qquad [D, K_\mu] = -\mathrm{i}K_\mu, \tag{5.1}$$
$$[M_{\mu\nu}, P_\rho] = \mathrm{i}(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu), \qquad [M_{\mu\nu}, K_\rho] = \mathrm{i}(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu), \tag{5.2}$$
$$[K_\mu, P_\nu] = 2\mathrm{i}(\eta_{\mu\nu}D - M_{\mu\nu}). \tag{5.3}$$

Changing from Hermitian to anti-Hermitian generators removes the explicit factors of i; this explains several sign differences among textbooks.

In radial quantization, a primary state $|\Delta, \ell\rangle$ obeys

$$K_\mu|\Delta, \ell\rangle = 0, \qquad D|\Delta, \ell\rangle = \mathrm{i}\Delta|\Delta, \ell\rangle$$

(up to the Hermiticity convention), and descendants are generated by arbitrary products of P_μ. Since the translations commute, the level-n descendants lie in

$$\mathrm{Sym}^n(\square) \otimes \ell.$$

This is the representation-theoretic origin of the momentum-generating function $P^{(d)}$ used below. For even $d = 2r$, an element of the vector torus has eigenvalues $(x_1, x_1^{-1}, \ldots, x_r, x_r^{-1})$, hence

$$P^{(2r)}(q; x) = \sum_{n=0}^{\infty} q^n \chi_{\mathrm{Sym}^n(\square)}(x) = \prod_{i=1}^{r} \frac{1}{(1 - qx_i)(1 - q/x_i)} = \frac{1}{\det_\square(1 - qg)}.$$

5.4 Unitarity bounds and shortening

For four-dimensional conformal primaries $[\Delta; (j_L, j_R)]$, the standard unitarity bounds are

$$\begin{cases} \Delta \geq j_L + j_R + 2, & j_L j_R \neq 0, \\ \Delta \geq j_L + j_R + 1, & \text{exactly one of } j_L, j_R \text{ is nonzero}, \\ \Delta \geq 1, & j_L = j_R = 0. \end{cases}$$

Saturation creates null descendants and thus shortened characters.

In general d, a scalar obeys

$$\Delta \geq \frac{d-2}{2},$$

a spinor obeys

$$\Delta \geq \frac{d-1}{2},$$

and a symmetric-traceless spin-$\ell \geq 1$ primary obeys

$$\Delta \geq \ell + d - 2.$$

At saturation, the latter is a conserved current.

For a free scalar at the bound, the descendant tower has the EOM subtraction

$$\chi_{\text{free scalar}}(q, z) = q^{(d-2)/2}(1 - q^2)P^{(d)}(q, z).$$

The factor $(1-q^2)$ follows directly from the symmetric-traceless decomposition

$$\chi_{\mathrm{ST}^n(\square)} = \chi_{\mathrm{Sym}^n(\square)} - \chi_{\mathrm{Sym}^{n-2}(\square)},$$

because the trace descendant is proportional to $\partial^2\phi$ and vanishes in the free conformal module.

For a conserved current J_μ with $\Delta = d-1$,

$$\partial^\mu J_\mu = 0$$

creates a scalar null at dimension d, hence

$$\chi_J = \chi_{[d-1;\square]} - \chi_{[d;\mathbf{1}]}.$$

In four dimensions, the self-dual field strength has the inclusion–exclusion character

$$\chi_{F_L} = D^2 P(D;\alpha,\beta)\left[\chi_1(\alpha) - D\chi_{1/2}(\alpha)\chi_{1/2}(\beta) + D^2\right],$$

with the anti-self-dual expression obtained by $\alpha \leftrightarrow \beta$. The first subtraction corresponds to the Maxwell/Bianchi null relation; the D^2 term restores the relation among those null descendants themselves.

5.5 Master Hilbert-series formula with derivatives

A convenient schematic form is

$$\boxed{H = \int \mathrm{d}\mu_{\mathrm{Lorentz}}\,\mathrm{d}\mu_{\mathrm{internal}}\,\frac{1}{P(D;x)}\;\mathrm{PE}\!\left[\sum_i \Phi_i\,\chi_i(D;x;g)\right] + \Delta H.}$$

Here the single-particle characters χ_i are already EOM-shortened. The factor $1/P$ removes generic total-derivative descendants. The additional term ΔH is an exceptional cohomological correction; omitting it is not always valid.

5.6 Standard Model representations

The gauge group is

$$G_{\mathrm{SM}} = SU(3)_c \times SU(2)_W \times U(1)_Y.$$

A quark doublet has

$$Q \sim (\mathbf{3},\mathbf{2})_{1/6}.$$

At the level of Lorentz representations,

$$H:(0,0),\quad \psi_L:(\tfrac{1}{2},0),\quad \psi_R:(0,\tfrac{1}{2}),\quad X_L:(1,0),\quad X_R:(0,1),\quad D_\mu:(\tfrac{1}{2},\tfrac{1}{2}).$$

When all fermions are written as left-handed Weyl fields, the conventional SMEFT variables u^c, d^c, e^c transform in conjugate gauge representations and carry opposite hypercharges relative to the corresponding physical right-handed fields. This convention must be fixed before constructing the PE.

A generic monomial

$$\phi^p \psi_L^{q_L} \psi_R^{q_R} D^r X_L^{s_L} X_R^{s_R}$$

has canonical dimension

$$d = p + \frac{3}{2}(q_L+q_R) + r + 2(s_L+s_R).$$

5.6.1 Single-particle PE arguments for the Standard Model

For a Hilbert-series implementation, each building block is assigned a field-counting spurion and multiplied by its Lorentz and gauge character. Schematically,

$$\begin{aligned}
H: &\quad H\,\chi_\phi(D;\alpha,\beta)\,\chi_{\mathbf{2}}^{SU(2)}(w)\,y^{1/2}, && (5.4)\\
H^\dagger: &\quad H^\dagger\,\chi_\phi(D;\alpha,\beta)\,\chi_{\mathbf{2}}^{SU(2)}(w)\,y^{-1/2}, && (5.5)\\
Q: &\quad Q\,\chi_{\psi_L}(D;\alpha,\beta)\,\chi_{\mathbf{3}}^{SU(3)}(c)\chi_{\mathbf{2}}^{SU(2)}(w)y^{1/6}, && (5.6)\\
L: &\quad L\,\chi_{\psi_L}(D;\alpha,\beta)\,\chi_{\mathbf{2}}^{SU(2)}(w)y^{-1/2}, && (5.7)\\
B_L: &\quad B_L\,\chi_{F_L}(D;\alpha,\beta), && (5.8)\\
W_L: &\quad W_L\,\chi_{F_L}(D;\alpha,\beta)\chi_{\mathbf{3}}^{SU(2)}(w), && (5.9)\\
G_L: &\quad G_L\,\chi_{F_L}(D;\alpha,\beta)\chi_{\mathbf{8}}^{SU(3)}(c), && (5.10)
\end{aligned}$$

with analogous right-chiral field strengths. Fermion entries go into PE_F; bosonic entries go into PE_B. Hypercharge singlets are projected by

$$\oint_{|y|=1}\frac{\mathrm{d}y}{2\pi\mathrm{i}y},$$

while the nonabelian singlets are projected by their Haar measures. This is the precise version of the field-by-field plethystic argument written on source pages 74–76.

5.7 Polynomial to Covariant form of operators

The monomials are reduced to the same basis classes after IBP, EOM, and algebraic identities.

5.7.1 Total derivatives

Examples such as $D^2\phi$ or ϕD^2 should first be embedded in a complete gauge/Lorentz singlet. A representative identity is

$$D_\mu\left(\phi^\dagger D^\mu\phi\right)=(D_\mu\phi)^\dagger D^\mu\phi+\phi^\dagger D^2\phi,$$

so inside the action

$$\phi^\dagger D^2\phi\simeq-(D_\mu\phi)^\dagger D^\mu\phi.$$

For field strengths,

$$D_\mu D_\nu X^{\mu\nu}=\frac{1}{2}[D_\mu,D_\nu]X^{\mu\nu},$$

because the symmetric derivative part contracts an antisymmetric tensor. The commutator then produces an additional field-strength insertion.

5.7.2 Fermion bilinears

The chirality-preserving vector currents are

$$\bar{\psi}_L\gamma^\mu\psi_L,\qquad\bar{\psi}_R\gamma^\mu\psi_R,$$

whereas scalar bilinears mix chiralities, such as $\bar{\psi}_L\psi_R$. Tensor bilinears are most transparently treated in two-component notation, where self-dual and anti-self-dual tensors couple to the corresponding chiral spinors.

A derivative acting on a fermion in an EFT monomial can be decomposed into a gamma-trace and gamma-traceless part. The gamma-trace is EOM-redundant at leading order because

$$\mathrm{i}\,\not{D}\psi = m\psi + \text{interactions}.$$

Similarly, the gauge EOM converts $D_\mu X^{\mu\nu}$ into matter currents. Thus structures denoted schematically by D^2X^2, $\psi^2\phi D^2$, or $\psi^2 XD$ in the source should be understood as starting points before basis reduction.

5.7.3 A representative IBP identity

For a fermion current times a scalar current,

$$D_\mu\left[(\bar{\psi}\gamma^\mu\psi)(\phi^\dagger\phi)\right] = (D_\mu\bar{\psi})\gamma^\mu\psi\,(\phi^\dagger\phi) + \bar{\psi}\gamma^\mu D_\mu\psi\,(\phi^\dagger\phi) + (\bar{\psi}\gamma^\mu\psi)D_\mu(\phi^\dagger\phi).$$

The integrated total derivative vanishes under the usual boundary conditions, establishing one linear relation among the three displayed operator structures. EOM may then be used as a *second*, conceptually distinct reduction.

5.8 Useful group and spinor identities

For Pauli matrices τ^I,

$$(\tau^I)_{ij}(\tau^I)_{kl} = 2\delta_{il}\delta_{jk} - \delta_{ij}\delta_{kl}.$$

For $SU(3)$ generators $T^A = \lambda^A/2$,

$$(T^A)_{ij}(T^A)_{kl} = \frac{1}{2}\delta_{il}\delta_{jk} - \frac{1}{6}\delta_{ij}\delta_{kl}.$$

With $\sigma^\mu = (1,\vec{\sigma})$ and $\bar{\sigma}^\mu = (1,-\vec{\sigma})$ in mostly-minus signature,

$$(\sigma^\mu)_{\alpha\dot{\alpha}}(\bar{\sigma}_\mu)^{\dot{\beta}\beta} = 2\delta_\alpha{}^\beta\delta_{\dot{\alpha}}{}^{\dot{\beta}}.$$

Signs in identities involving two undotted σ matrices depend on the ϵ-tensor conventions and should not be mixed across sources.

For a single commuting Higgs doublet,

$$(H^\dagger\tau^I H)(H^\dagger\tau^I H) = (H^\dagger H)^2.$$

5.9 Dimension-six SMEFT operator classes

A useful class decomposition is

$$X^3,\quad H^6,\quad H^4D^2,\quad X^2H^2,\quad \psi^2H^3,\quad \psi^2XH,\quad \psi^2H^2D,\quad \psi^4.$$

These are classes, not individual operators. "Warsaw basis" denotes a particular nonredundant basis choice; "SILH" denotes a different organization/basis adapted to certain strongly-interacting-Higgs scenarios, not an extra unique operator class.

Chapter 6

PDE preliminaries, semigroups, and integral transforms

6.1 First-order flows

Given a time-independent vector field $\xi = \xi^\mu \partial_\mu$ and generator $L = \xi^\mu \partial_\mu$, the formal solution of

$$(\partial_t - L)u(t, x) = 0, \qquad u(0, x) = f(x)$$

is

$$u(t) = \mathrm{e}^{tL} f,$$

provided L generates an appropriate flow/semigroup on the chosen function space.

6.2 Second-order operators and ellipticity

For

$$A = a^{\mu\nu}(x)\partial_\mu\partial_\nu + b^\mu(x)\partial_\mu + c(x),$$

the PDE type is determined by the principal symbol

$$\sigma_2(A)(x, \xi) = a^{\mu\nu}(x)\xi_\mu\xi_\nu.$$

A real symmetric principal part is elliptic when this quadratic form is positive (or negative) definite for every $\xi \neq 0$. Uniform ellipticity strengthens this to a bound

$$a^{\mu\nu}\xi_\mu\xi_\nu \geq c|\xi|^2, \qquad c > 0.$$

Lower-derivative terms do not determine elliptic/hyperbolic/parabolic type.

6.3 Heat semigroups

If $A \geq 0$ is self-adjoint, then

$$\mathrm{e}^{-tA}, \qquad t \geq 0,$$

is a contraction semigroup. In general it need not extend to a bounded group for negative t; saying that the inverse is simply “undefined” is too strong, because its existence depends on spectrum, domain, and function space.

For discrete spectrum $A\phi_n = \lambda_n \phi_n$,

$$K(t; x, y) = \sum_n e^{-t\lambda_n} \phi_n(x)\phi_n^\dagger(y).$$

For a complex self-adjoint problem,

$$K(t; x, y) = K(t; y, x)^\dagger,$$

not necessarily $K(t; x, y) = K(t; y, x)$ without a reality condition. If the heat operator is trace class,

$$\operatorname{Tr} e^{-tA} = \int d\mu(x)\ \operatorname{tr} K(t; x, x).$$

6.4 Laplace and Mellin transforms

The Laplace transform and Bromwich inverse are

$$F(s) = \int_0^\infty dt\, e^{-st} f(t), \qquad f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} ds\, e^{st} F(s).$$

The Mellin transform is

$$\mathcal{M}[f](s) = \int_0^\infty t^{s-1} f(t)\, dt,$$

with inverse along a vertical line inside the fundamental strip. Under $t = e^x$, it becomes a Fourier transform in x up to a simple exponential weight.

The gamma function has simple poles at $s = -n$, $n = 0, 1, 2, \ldots$, with

$$\operatorname{Res}_{s=-n} \Gamma(s) = \frac{(-1)^n}{n!}.$$

Poles of a Mellin transform encode asymptotic powers (and multiple poles encode logarithms). The resulting small-t series is generically *asymptotic*; analyticity and remainder control are required before claiming convergence.

6.5 What the mass exponential suppresses

In a proper-time integral, $e^{-M^2 t}$ suppresses the *large-t* region. UV divergences arise from $t \to 0$, so the mass factor is not by itself a UV regulator. At fixed positive t, e^{-tp^2} suppresses large momentum, and its Fourier transform gives the familiar Gaussian in coordinate space.

Chapter 7

Heat kernels and the one-loop effective action

7.1 From a general second-order elliptic operator to Laplace type

The source begins from a local second-order operator

$$L = -a^{ij}(x)\partial_i\partial_j + b^i(x)\partial_i + c(x).$$

If $a^{ij} = g^{ij}$ is the inverse of a Riemannian metric, the first-derivative term can be absorbed into a bundle connection. One then rewrites the operator in Laplace form

$$L = -g^{-1/2}(\partial_i + \mathcal{A}_i)\, g^{1/2} g^{ij}(\partial_j + \mathcal{A}_j) + \mathcal{Q}.$$

For constant g^{ij} this reduces to

$$L = -g^{ij}(\partial_i + \mathcal{A}_i)(\partial_j + \mathcal{A}_j) + \mathcal{Q}.$$

This completion is the geometric reason why a generic elliptic quadratic fluctuation operator can be handled by heat-kernel methods once its principal symbol is of Laplace type.

For a constant abelian connection $\mathcal{A}_i$ commuting with $\mathcal{Q}$, the kernel is simply the free Gaussian multiplied by a parallel-transport factor and $e^{-t\mathcal{Q}}$. For noncommuting or space-dependent backgrounds, path ordering/parallel transport replaces that elementary exponential.

7.2 Laplace-type operator and free kernel

We now adopt the consistent Euclidean convention

$$\boxed{\Delta = -\mathcal{D}^2 + M^2 + U}, \qquad \Omega_{\mu\nu} = [\mathcal{D}_\mu, \mathcal{D}_\nu].$$

The heat kernel solves

$$(\partial_t + \Delta_x)K(t;x,y) = 0, \qquad K(0;x,y) = \delta(x-y)\mathbf{1}.$$

In flat space with $U = 0$ and trivial connection,

$$K_0(t;x,y) = \frac{1}{(4\pi t)^{d/2}} \exp\left[-\frac{(x-y)^2}{4t} - M^2 t\right] \mathbf{1}.$$

7.3 Small-t expansion

Factor the free kernel,

$$K(t;x,y) = K_0(t;x,y)H(t;x,y),$$

and use the asymptotic expansion

$$H(t;x,y) \sim \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} b_k(x,y), \qquad t \to 0^+.$$

It is a short-time asymptotic series; it is not justified by large-t convergence.

If one temporarily uses the source-paper derivative notation D_μ and $z = x - y$, the recursion is

$$(k + z \cdot D)b_k = k(D^2 + U)b_{k-1},$$

with the understanding that the sign of D^2 is tied to the paper's convention. The invariant coincidence coefficients below are conventionally safer than manipulating this recursion without fixing signs.

7.4 Recursion relation used in the handwritten and typed derivations

With $z^\mu = (x-y)^\mu$ and in the source derivative convention, substitution of

$$K = K_0 \sum_{k \geq 0} \frac{(-t)^k}{k!} b_k$$

into the heat equation gives

$$\boxed{(k + z \cdot D)b_k = k(D^2 + U)b_{k-1}}.$$

After applying m covariant derivatives and taking $z \to 0$,

$$D_{\mu_1} \cdots D_{\mu_m} b_k\big|_{z=0} = \frac{1}{m+k} \left\{ k D_{\mu_1} \cdots D_{\mu_m}(D^2 + U)b_{k-1} - T_{\mu_1 \cdots \mu_m} b_k \right\}\Big|_{z=0},$$

where

$$T_{\mu_1 \cdots \mu_m} = D_{\mu_1} \cdots D_{\mu_m}(z \cdot D)\big|_{z=0} - m D_{\mu_1} \cdots D_{\mu_m}.$$

The useful recurrence

$$T_{\mu_1 \cdots \mu_m} = D_{\mu_1} T_{\mu_2 \cdots \mu_m} + R_{\mu_2 \cdots \mu_m, \mu_1}, \qquad R_{\mu_2 \cdots \mu_m, \mu_1} = [D_{\mu_2} \cdots D_{\mu_m}, D_{\mu_1}],$$

reduces every T to commutators $G_{\mu\nu} = [D_\mu, D_\nu]$ and their covariant derivatives. In particular $T_\mu = T_{\mu\mu} = 0$ and

$$T_{\mu\mu\nu\nu} = -2G_{\mu\nu}G_{\mu\nu} - 2J_\nu D_\nu, \qquad J_\nu = D_\mu G_{\mu\nu},$$

with the sign tied to the source's D convention. This is the computational engine behind the b_1, b_2, b_3 pages.

7.5 Coincidence coefficients

For flat spacetime, with $G_{\mu\nu} = [D_\mu, D_\nu]$, $J_\nu = D_\mu G_{\mu\nu}$, and semicolon denoting covariant differentiation, the traced/coincidence structures quoted in the source paper and checked against the inserted typed pages are

$$\mathrm{tr}[b_0] = \mathrm{tr}\,\mathbf{1}, \tag{7.1}$$

$$\mathrm{tr}[b_1] = \mathrm{tr}\,U, \tag{7.2}$$

$$\mathrm{tr}[b_2] = \mathrm{tr}\left[U^2 + \frac{1}{6}G_{\mu\nu}G_{\mu\nu}\right], \tag{7.3}$$

$$\mathrm{tr}[b_3] = \mathrm{tr}\left[U^3 - \frac{1}{2}(U_{;\mu})^2 + \frac{1}{2}UG_{\mu\nu}G_{\mu\nu} - \frac{1}{10}J_\nu J_\nu + \frac{1}{15}G_{\mu\nu}G_{\nu\rho}G_{\rho\mu}\right]. \tag{7.4}$$

These are reduced under the trace/integral; total derivatives have been removed. The unreduced local coefficient contains additional derivative terms such as $U_{;\mu\mu}$.

For example, the local second coefficient before discarding a total derivative is

$$[b_2] = U^2 + \frac{1}{6}G_{\mu\nu}G_{\mu\nu} + \frac{1}{3}U_{;\mu\mu}.$$

7.5.1 The longer b_4 structure

The inserted page also records the following reduced b_4 expression:

$$\mathrm{tr}[b_4] = \mathrm{tr}\Big[U^4 + U^2U_{;\mu\mu} + \frac{4}{5}U^2(G_{\mu\nu})^2 + \frac{1}{5}(UG_{\mu\nu})^2 + \frac{1}{5}(U_{;\mu\mu})^2 - \frac{2}{5}UU_{;\nu}J_\nu - \frac{2}{5}U(J_\mu)^2 \tag{7.5}$$

$$+ \frac{2}{15}U_{;\mu\mu}(G_{\rho\sigma})^2 + \frac{4}{15}UG_{\mu\nu}G_{\nu\rho}G_{\rho\mu} + \frac{8}{15}U_{;\nu\mu}G_{\rho\mu}G_{\rho\nu} + \frac{1}{35}(J_{\mu;\nu})^2 \tag{7.6}$$

$$+ \frac{16}{105}G_{\mu\nu}J_\mu J_\nu + \frac{1}{420}(G_{\mu\nu}G_{\rho\sigma})^2 + \frac{17}{210}(G_{\mu\nu})^2(G_{\rho\sigma})^2 \tag{7.7}$$

$$+ \frac{1}{105}G_{\mu\nu}G_{\nu\rho}G_{\rho\sigma}G_{\sigma\mu} + \frac{2}{35}(G_{\mu\nu}G_{\nu\rho})^2 + \frac{16}{105}J_{\nu;\mu}G_{\nu\sigma}G_{\sigma\mu}\Big]. \tag{7.8}$$

For matrix-valued U and G, products are ordered as written inside the trace.

For completeness, the selected terms written on source page 126 are transcribed below. The vertical bar indicates the subset of operator classes retained in that source:

$$\mathrm{tr}[b_5]_{U^5,U^4,U^3,U^2} = \mathrm{tr}\Big[U^5 + 2U^3U_{;\mu\mu} + U^2(U_{;\mu})^2 + U^3G^2 + \frac{2}{3}U^2G_{\mu\nu}UG_{\mu\nu} \tag{7.9}$$

$$- \frac{1}{3}U^2U_{;\mu}J_\mu + \frac{1}{3}U^2J_\mu U_{;\mu} + \frac{1}{3}UG_{\mu\nu}U_{;\mu}U_{;\nu} + \frac{1}{3}UU_{;\mu}U_{;\nu}G_{\mu\nu} \tag{7.10}$$

$$+ UU_{;\mu\mu}U_{;\nu\nu} + \frac{2}{3}U_{;\mu\mu}(U_{;\nu})^2 + \mathcal{O}(D^6U^2)\Big], \tag{7.11}$$

where $G^2 \equiv G_{\mu\nu}G_{\mu\nu}$. The selected b_6 terms are

$$\mathrm{tr}[b_6]_{U^6,U^5,U^4} = \mathrm{tr}\Big[U^6 + 3U^4U_{;\mu\mu} + 2U^3(U_{;\mu})^2 + \frac{12}{7}U^2U_{;\nu\mu}U_{;\mu\nu} + \frac{17}{14}(U_{;\mu}U_{;\nu})^2 \tag{7.12}$$

$$+ \frac{9}{7}UU_{;\nu\mu}UU_{;\mu\nu} + \frac{26}{7}U_{;\nu\mu}U_{;\mu}U_{;\nu}U + \frac{18}{7}U_{;\nu\mu}U_{;\mu}UU_{;\nu} \tag{7.13}$$

$$+ \frac{26}{7}U_{;\nu\mu}UU_{;\mu}U_{;\nu} + \frac{9}{7}(U_{;\mu})^2(U_{;\nu})^2 + \frac{18}{7}G_{\mu\nu}UU_{;\mu}U_{;\nu}U \tag{7.14}$$

$$+ \frac{5}{7}U^4G^2 + \frac{8}{7}U^3G_{\mu\nu}UG_{\mu\nu} + \frac{18}{7}G_{\mu\nu}U_{;\mu}U^2U_{;\nu} + \frac{9}{14}(U^2G_{\mu\nu})^2 \tag{7.15}$$

$$+\frac{26}{7}G_{\mu\nu}U_{;\mu}UU_{;\nu}U+\frac{8}{7}G_{\mu\nu}UU_{;\mu}UU_{;\nu}+\frac{24}{7}G_{\mu\nu}U_{;\mu}U_{;\nu}U^2-\frac{2}{7}G_{\mu\nu}U^2U_{;\mu}U_{;\nu}\Big]. \tag{7.16}$$

Finally,

$$\mathrm{tr}[b_7]_{U^7,U^6}=\mathrm{tr}\left[U^7-5U^4(U_{;\mu})^2-8U^3U_{;\mu}UU_{;\mu}-\frac{9}{2}(U^2U_{;\mu})^2\right],\qquad \mathrm{tr}[b_8]_{U^8}=\mathrm{tr}[U^8].$$

These last three displays should not be mistaken for complete b_5, b_6, b_7, b_8 coefficients.

7.6 What may be used in deriving the coefficients

The heat-kernel recursion is an off-shell statement. One may use algebraic commutator identities and Bianchi/Jacobi identities freely. IBP is permitted only after passing to the *integrated traced* coefficient. Classical EOM should not be used to derive universal heat-kernel coefficients.

7.7 One-loop determinant and proper time

For a real bosonic fluctuation with quadratic operator Δ,

$$\Gamma^{(1)}=\frac{1}{2}\,\mathrm{Tr}\ln\Delta.$$

For a complex scalar the prefactor is 1, and for a Grassmann field the sign is reversed. The formal identity

$$\ln\lambda=-\int_0^\infty\frac{\mathrm{d}t}{t}\,\mathrm{e}^{-t\lambda}$$

is divergent by an additive constant. A precise form is, for example,

$$\ln\frac{\lambda}{\mu^2}=-\int_0^\infty\frac{\mathrm{d}t}{t}\left(\mathrm{e}^{-t\lambda}-\mathrm{e}^{-t\mu^2}\right).$$

Using the heat-kernel expansion in d dimensions gives

$$\mathcal{L}^{(1)}_{\mathrm{eff}}=\frac{c_s}{(4\pi)^{d/2}}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}M^{d-2k}\Gamma\Big(k-\frac{d}{2}\Big)\,\mathrm{tr}[b_k],$$

where $c_s=1/2$ for a real scalar and $c_s=1$ for a complex scalar.

In $d=4-\epsilon$, the poles for $k=0,1,2$ renormalize operators of dimension at most four. In an $\overline{\mathrm{MS}}$-type convention the finite terms have the familiar schematic dependence

$$M^4\left(\ln\frac{M^2}{\mu^2}-\frac{3}{2}\right)b_0,\quad M^2\left(1-\ln\frac{M^2}{\mu^2}\right)b_1,\quad \ln\frac{M^2}{\mu^2}\,b_2,$$

with overall factors determined by the field multiplicity and the precise subtraction convention.

Chapter 8

Fermions, bosonization, and nondegenerate heavy masses

8.1 Bosonizing a fermionic determinant

For a Dirac operator coupled to scalar/pseudoscalar backgrounds, the source notes follow the standard strategy of multiplying first-order operators to obtain a Laplace-type problem. In the conventions of the cited dimension-eight fermion calculation, one may write

$$\mathcal{L}_{\text{eff}}^{\Psi} = -\frac{\mathrm{i}}{2}\,\mathrm{tr}\ln\left[-\widetilde{P}^2 + M_f^2 + U_f\right],$$

with

$$U_f = Y + 2M_f\Sigma, \tag{8.1}$$
$$\Sigma = S + \mathrm{i}\gamma_5 R, \tag{8.2}$$
$$Y = -\frac{1}{2}\sigma^{\mu\nu}F_{\mu\nu} + S^2 + 3R^2 - (\not{P}S) + \mathrm{i}\gamma_5(RS + SR), \tag{8.3}$$
$$\widetilde{P}_\mu = P_\mu - \mathrm{i}\gamma_5\gamma_\mu R, \tag{8.4}$$
$$\widetilde{G}_{\mu\nu} = F_{\mu\nu} + \Gamma_{\mu\nu}, \tag{8.5}$$
$$\Gamma_{\mu\nu} = \mathrm{i}\gamma_5\gamma_\mu(P_\nu R) - \mathrm{i}\gamma_5\gamma_\nu(P_\mu R) + 2\sigma_{\mu\nu}R^2. \tag{8.6}$$

The exact signs here are convention-dependent; the displayed set is internally tied to the source paper's definitions.

8.2 Multiplicative anomaly

For zeta-regularized determinants, $\ln\det(AB)$ need not equal $\ln\det A + \ln\det B$ without an anomaly term. In the source paper's even-d construction,

$$\mathcal{A} = 2\sum_{j=1}^{d/2}\frac{(-1)^j M_f^{2j}}{j!}Q_j[b_{d/2-j}], \qquad Q_j = \sum_{\ell=1}^{j}\frac{1}{2\ell-1}.$$

For $d = 4$,

$$\boxed{\mathcal{A} = -2M_f^2[b_1] + \frac{4}{3}M_f^4[b_0].}$$

Thus, in this setup the anomaly affects only low-dimensional (renormalizable) structures, not the genuinely higher-dimensional EFT terms.

8.3 Nondegenerate heavy masses

If the mass matrix does not commute with the interaction,

$$[M^2, U] \neq 0,$$

one cannot pull e^{-tM^2} through the interaction. Introduce an interaction-picture operator

$$A_I(s) = e^{sM^2} A e^{-sM^2}$$

and use the Dyson expansion

$$e^{-t(M^2+A)} = e^{-tM^2}\, \mathcal{T} \exp\left[-\int_0^t \mathrm{d}s\, A_I(s)\right].$$

Matrix elements then generate simplex integrals and divided-difference loop functions. More explicitly, with $s_i = t\alpha_i$ and $0 < \alpha_n < \cdots < \alpha_1 < 1$,

$$\mathcal{T} \exp\left[-\int_0^t \mathrm{d}s\, A_I(s)\right] = \mathbf{1} + \sum_{n=1}^{\infty} (-1)^n \int_0^t \mathrm{d}s_1 \int_0^{s_1} \mathrm{d}s_2 \cdots \int_0^{s_{n-1}} \mathrm{d}s_n\, A_I(s_1) \cdots A_I(s_n).$$

For a diagonal mass matrix, each matrix element carries exponentials such as

$$(A_I(s))_{ij} = e^{s(M_i^2 - M_j^2)} A_{ij},$$

which is why the proper-time integrations produce ratios of mass differences and logarithms. This also explains why apparently singular denominators have finite coincident-mass limits.

For two masses it is convenient to define

$$\Delta_{12}^2 \equiv M_1^2 - M_2^2,$$

where the superscript on Δ_{12}^2 is part of the notation for a difference of squared masses, not a further square. A coefficient displayed in the source notes is

$$C_{12} = 1 - \frac{M_1^2 \ln(M_1^2/\mu^2)}{\Delta_{12}^2} + \frac{M_2^2 \ln(M_2^2/\mu^2)}{\Delta_{12}^2},$$

for a quadratic structure $C_{ij}\, \mathrm{tr}(U_{ij} U_{ji})$. A cubic example is

$$C_{122} = \frac{1}{\Delta_{12}^2} - \frac{M_1^2 \ln(M_1^2/M_2^2)}{(\Delta_{12}^2)^2}.$$

Such expressions are smooth in the degenerate limit after the appropriate divided-difference limit is taken; one should not substitute $M_1 = M_2$ before taking that limit.

8.4 Matching and the hard region

A local Wilson coefficient is obtained from the hard part of the UV theory after subtracting EFT contributions that reproduce the same infrared physics. In diagrammatic language this is the method-of-regions/matching statement; in functional methods the same separation appears in the heavy/light fluctuation operator. Individual hard and soft pieces can be scheme-dependent, whereas matched physical amplitudes are not.

Chapter 9

Background-field example: massive scalar QED

Source-note pages: 137–141; pages 142–146 are blank.

A representative starting Lagrangian in the source notes is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M_A^2 A_\mu A^\mu - \frac{1}{2\xi}(\partial \cdot A)^2 + (D_\mu\Phi)^\dagger D^\mu\Phi - m^2\Phi^\dagger\Phi - \frac{\lambda}{6}(\Phi^\dagger\Phi)^2,$$

with $D_\mu = \partial_\mu - \mathrm{i}eA_\mu$. Write

$$\Phi = \frac{1}{\sqrt{2}}(\phi_1 + \mathrm{i}\phi_2)$$

and split every field into background plus fluctuation,

$$\phi_i = \bar{\phi}_i + \eta_i, \qquad A_\mu = \bar{A}_\mu + a_\mu.$$

The quadratic action is

$$S^{(2)} = \frac{1}{2}\int \mathrm{d}^4x \begin{pmatrix} \eta_1 & \eta_2 & a_\mu \end{pmatrix} \begin{pmatrix} \Delta_{11} & \Delta_{12} & \Delta_{1\nu} \\ \Delta_{21} & \Delta_{22} & \Delta_{2\nu} \\ \Delta_{\mu 1} & \Delta_{\mu 2} & \Delta_{\mu\nu} \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ a_\nu \end{pmatrix}.$$

The entries follow from the Hessian

$$\Delta_{AB}(x,y) = \left.\frac{\delta^2 S}{\delta\varphi_A(x)\delta\varphi_B(y)}\right|_{\bar{\varphi}}.$$

A judicious background-field gauge can remove first-derivative scalar–vector mixing and put the Hessian into Laplace type, after which the heat-kernel machinery applies directly. The source ends this segment with the exercise of extracting the one-loop effective Lagrangian through dimension six.

Chapter 10

Green functions, short-distance singularities, and multiloop structure

10.1 Green function from the heat kernel

For positive Δ,

$$G(x,y) = \Delta^{-1}(x,y) = \int_0^\infty \mathrm{d}t\, K(t;x,y).$$

Using the heat-kernel series yields

$$G(x,y) \sim \sum_{n=0}^{\infty} g_n(z)\, \widetilde{b}_n(x,y), \qquad z = x-y,$$

where

$$g_n(z) = \frac{(-1)^n}{n!(4\pi)^{d/2}} \int_0^\infty \mathrm{d}t\, t^{n-d/2} \exp\left[-\frac{z^2}{4t} - M^2 t\right] \tag{10.1}$$

$$= \frac{(-1)^n 2}{n!(4\pi)^{d/2}} \left(\frac{|z|}{2M}\right)^{n+1-d/2} K_{n+1-d/2}(M|z|). \tag{10.2}$$

At coincidence, dimensional regularization gives

$$g_n(0) = \frac{(-1)^n}{n!(4\pi)^{d/2}} \Gamma\left(n+1-\frac{d}{2}\right) M^{d-2n-2}.$$

The small-z singularities follow from the small-argument expansion of the modified Bessel function K_ν.

10.2 Local distributional poles

With the Fourier convention

$$\widetilde{f}(k) = \int \mathrm{d}^d z\, \mathrm{e}^{\mathrm{i}k\cdot z} f(z),$$

one has, by analytic continuation,

$$\int \mathrm{d}^d z\, \frac{\mathrm{e}^{\mathrm{i}k\cdot z}}{(z^2)^a} = 2^{d-2a} \pi^{d/2} \frac{\Gamma(d/2-a)}{\Gamma(a)} (k^2)^{a-d/2}.$$

Set

$$a = \frac{d}{2} + n + \zeta, \qquad n = 0,1,2,\ldots.$$

Then the pole is local:

$$\frac{1}{(z^2)^{d/2+n+\zeta}} = \frac{(-1)^{n+1}\pi^{d/2}}{4^n n!\Gamma(d/2+n)}\frac{1}{\zeta}(-\partial^2)^n\delta^{(d)}(z) + \text{finite},$$

for the stated Fourier convention.

Numerator factors can be generated by differentiation. For example,

$$\frac{z_\mu}{(z^2)^{a+1}} = -\frac{1}{2a}\partial_\mu\frac{1}{(z^2)^a},$$

which is a safer way to derive tensor distributions than memorizing convention-sensitive formulas.

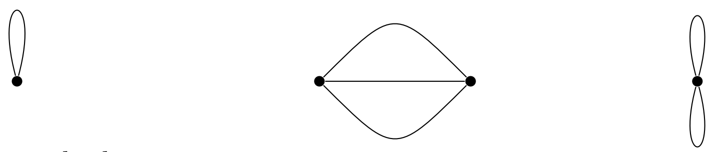
10.3 Loop topology counting

For a connected vacuum graph with V vertices of valences m_i and I internal lines,

$$I = \frac{1}{2}\sum_{i=1}^{V} m_i, \qquad L = I - V + 1 = 1 + \frac{1}{2}\sum_i m_i - V.$$

At two loops, the familiar scalar vacuum topologies are the sunset/theta graph and the figure-eight graph. At three loops, additional graph-isomorphism classes arise, and identifying insertion points in a multi-vertex skeleton produces the degenerations sketched on source pages 163–164.

10.4 Schematic vacuum topologies



one-loop tadpole

two-loop sunset/theta

two-loop figure-eight

The numerical symmetry factor of a vacuum graph depends on the normalization of the interaction vertices. It should be derived from the expansion of $\exp(-S_{\text{int}})$ rather than attached to a topology without specifying the Lagrangian convention.

Appendix A

Reconstruction of the typed b_1, b_2, b_3 calculation

The typed insertion on source pages 119–125 derives coincidence coefficients using a recurrence. The clean way to retain its pedagogical content is to separate the *local* coefficient from the *integrated traced* coefficient.

At coincidence,

$$[b_0] = \mathbf{1}, \qquad [b_1] = U,$$

and

$$[b_2] = U^2 + \frac{1}{6}G_{\mu\nu}G_{\mu\nu} + \frac{1}{3}U_{;\mu\mu}.$$

After integration and trace, the last term is a total derivative (under standard boundary assumptions), giving

$$\int \mathrm{d}^d x\ \mathrm{tr}[b_2] = \int \mathrm{d}^d x\ \mathrm{tr}\left(U^2 + \frac{1}{6}G^2\right).$$

At the next order, the local expression contains several total derivatives. After cyclic trace identities, Bianchi/Jacobi identities, and IBP, one reaches

$$\int \mathrm{d}^d x\ \mathrm{tr}[b_3] = \int \mathrm{d}^d x\ \mathrm{tr}\left[U^3 - \frac{1}{2}(U_{;\mu})^2 + \frac{1}{2}UG^2 - \frac{1}{10}J^2 + \frac{1}{15}G_{\mu\nu}G_{\nu\rho}G_{\rho\mu}\right].$$

No classical field equation is required.

Appendix B

Quick-reference formula sheet

$$
\begin{aligned}
[S] &= 0, \quad [\mathcal{L}] = d, \quad [\phi] = (d-2)/2, \quad [\psi] = (d-1)/2, \\
\mathrm{PE}_B[f] &= \exp \sum_{r \geq 1} \frac{f(x^r)}{r}, \qquad \mathrm{PE}_F[f] = \exp \sum_{r \geq 1} \frac{(-1)^{r+1} f(x^r)}{r}, \\
\chi_\lambda^{SU(N)} &= \frac{\det(\epsilon_i^{\lambda_j + N - j})}{\det(\epsilon_i^{N-j})}, \\
P(D; \alpha, \beta) &= \prod_{w \in \{\alpha\beta, \alpha/\beta, \beta/\alpha, (\alpha\beta)^{-1}\}} (1 - Dw)^{-1}, \\
\Delta_{\mathrm{HK}} &= -\mathcal{D}^2 + M^2 + U, \qquad K_0 = (4\pi t)^{-d/2} \mathrm{e}^{-(x-y)^2/4t - M^2 t}, \\
\mathcal{L}_{\text{eff}}^{(1)} &= \frac{c_s}{(4\pi)^{d/2}} \sum_{k \geq 0} \frac{(-1)^k}{k!} M^{d-2k} \Gamma(k - d/2) \operatorname{tr}[b_k], \\
G(x, y) &= \int_0^\infty \mathrm{d}t \, K(t; x, y), \\
L_{\text{graph}} &= I - V + 1.
\end{aligned}
$$

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